ENGINEERING CURRICULUM – 2023 ***B.Tech. R23 Regulations***

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Course Code 23HBS9910	COMPLEX VARIABLES & NUMERICAL METHODS	Credits 3-0-0:3

Course Outcomes:

CO	Statements	Blooms Level
CO1	Analyze limit, continuity and differentiation of functions of complex variables and Understand Cauchy-Riemann equations, analytic functions and various properties of analytic functions.	L2, L3
CO2	Understand Cauchy theorem, Cauchy integral formulas and apply these to evaluate complex contour integrals. Classify singularities and poles; find residues and evaluate complex integrals using the residue theorem	L3, L5
CO3	Apply numerical methods to solve algebraic and transcendental equations	L3
CO4	Derive interpolating polynomials using interpolation formulae	L2, L3
CO5	Solve differential and integral equations numerically	L3, L5

UNIT-I

Complex Variable – Differentiation

Introduction to functions of complex variable-concept of Limit & continuity- Differentiation, Cauchy-Riemann equations, analytic functions, harmonic functions, finding harmonic conjugate-construction of analytic function by Milne Thomson method.

UNIT-II

Complex Variable – Integration

Line integral-Contour integration, Cauchy's integral theorem (Simple Case), Cauchy Integral formula, Power series expansions: Taylor's series, zeros of analytic functions, singularities, Laurent's series, Residues, Cauchy Residue theorem (without proof), Evaluation of definite integral involving sine and cosine.

UNIT-III

Solution of Algebraic & Transcendental Equations

Introduction - Bisection Method, Regula-falsi method and Newton-Raphson method. Solution to system of Nonlinear equations: The method of iteration and Newton -Raphson method.

UNIT-IV

Interpolation

Finite differences-Newton's forward and backward interpolation formulae – Lagrange's formulae.
Curve fitting: Fitting of straight line, second-degree and Exponential curve by method of least squares.

UNIT-V

Solution of Initial value problems to Ordinary differential equations

Numerical solution of Ordinary Differential equations: Solution by Taylor's series-Picard's Method of successive Approximations-Euler's and modified Euler's methods-Runge-Kutta methods (second and fourth order).

Textbooks:

1. B.S.Grewal, Higher Engineering Mathematics, KhannaPublishers,2017, 44th Edition
2. S. S.Sastry, Introductory Methods of Numerical Analysis, PHI Learning Private Limited

Reference Books:

1. Erwin Kreyszig, Advanced Engineering Mathematics, John Wiley & Sons, 2018, 10th Edition.
2. B.V.Ramana, Higher Engineering Mathematics, by McGraw Hill publishers
3. R.K.Jain and S.R.K.Iyengar, Advanced Engineering Mathematics, Alpha Science International Ltd., 2021 5th Edition(9th reprint).

Web Resources:

1. https://onlinecourses.nptel.ac.in/noc17_ma14/preview
2. https://onlinecourses.nptel.ac.in/noc20_ma50/preview
3. http://nptel.ac.in/courses/111105090

PART –A

COMPLEX ANALYSIS

Unit-I

COMPLEX VARIABLE

1. 1 Algebraic Preliminaries: We shall recall some of the properties of a complex number.

(i) A complex number is of the form $z = a + ib$ where a and b are real numbers and i is the imaginary unit defined by $i = \sqrt{-1}$, a is called the real part of z and this is written as $R(z) = a$, b is called the imaginary part of z and this is written as $I(z) = b$.

(ii) If the two complex numbers $a + ib$ and $c + id$ are equal, then $a = c$ and $b = d$, i.e., the real and the imaginary parts of the first are respectively equal to the real and the imaginary parts of the second.

(iii) Complex numbers are assumed to obey the addition, subtraction, multiplication Division laws of Algebra. Thus,

$$(a + ib) + (c + id) = (a + c) + i(b + d)$$

$$(a + ib) - (c + id) = (a - c) + i(b - d)$$

$$(a + ib)(c + id) = (ac - bd) + i(ad + bc) \text{ since } i^2 = -1$$

$$\frac{(a + ib)}{(c + id)} = \frac{(a + ib)(c - id)}{(c + id)(c - id)}$$

$$= \left(\frac{ac + bd}{c^2 + d^2}\right) + i\left(\frac{bc - ad}{c^2 + d^2}\right)$$

(iv) Of the two complex numbers $a + ib$ and $a - ib$, each is said to be the conjugate of the other. The conjugate of a complex number z is usually written as $\bar{z}$. Sometimes $\bar{z}$ is also denoted by z^*.

If $z = (a + ib)$, then $\bar{z} = a - ib$.

$z\,\bar{z} = (a + ib)(a - ib) = a^2 + b^2$ which is purely real.

Also $\dfrac{z + \bar{z}}{2} = a =$ real part of $z = R(z)$

and $\dfrac{z - \bar{z}}{2} = b =$ imaginary part of $z = I(z)$

(v) The complex number $a + ib$ can be represented by a point in a plane referred to a set of rectangular x and y-axes such that the real part a represents the abscissa and the imaginary part b represents the ordinate of the point. In this manner, there is a one-to-one correspondence between the pair of real numbers (a, b) and the single complex number $a + ib$. In this case, the xy-plane is called the plane of a complex variable or the complex plane, the x-axis is called the real axis and the y-axis, the imaginary axis. Let the polar coordinates of the point (a, b) be (r, θ).

Then, $a = r\,cos\theta\ and\ b = r\,sin\theta$

So $r = \sqrt{a^2+b^2}\ and\ \theta = tan^{-1}\frac{b}{a}$

The number r is called the modulus and θ is called the amplitude or argument of the complex number $z = a + ib$. In symbols, we write

$$r = |z| = |a+ib| = \sqrt{a^2+b^2}$$

$$\theta = amp\,z = \arg z = tan^{-1}\frac{b}{a}$$

Now $z = a + ib = r(cos\theta + i\,sin\theta) = re^{i\theta}$

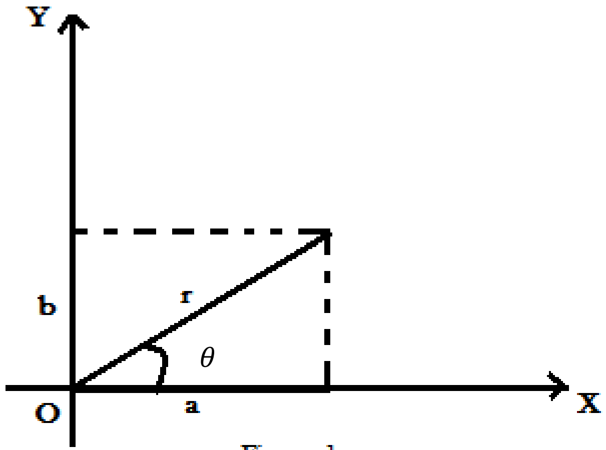


Figure 1

(vi) From the above polar mode of representation of a complex number, the rules for the product and quotient of two complex numbers follow immediately.

Thus, $z_1z_2 = (a_1+ib_1)(a_2+ib_2) = r_1e^{i\theta_1}.r_2e^{i\theta_2} = r_1r_2e^{i(\theta_1+\theta_2)}$

Hence, $|z_1z_2| = r_1r_2 = |z_1|.|z_2|$

and $\arg(z_{1,}z_2) = \theta_1 + \theta_2 = \arg z_1 + \arg z_2$

i.e., the modulus of the product is equal to the product of the modulus and the argument of the product is equal to the sum of the arguments.

$$\frac{z_1}{z_2} = \frac{r_1e^{i\theta_1}}{r_2e^{i\theta_2}} = \frac{r_1}{r_2}e^{i(\theta_1-\theta_2)}$$

So $\left|\frac{z_1}{z_2}\right| = \frac{r_1}{r_2} = \frac{|z_1|}{|z_2|}$ and $\arg\left(\frac{z_1}{z_2}\right) = \theta_1 - \theta_2 = \arg z_1 - \arg z_2$

i.e., the modulus of the quotient is the quotient of the modulus and the argument of the quotient is equal to the difference of the argument of the denominator from that of the numerator.

(vii) When n is positive integer,

$$z^n = \left(re^{i\theta}\right)^n = r^ne^{in\theta} = r^n(cosn\theta + i\,sinn\theta)$$

$$i.e., [r(cos\theta + i\ sin\theta)]^n = r^n(cosn\theta + i\ sinn\theta)$$

Hence $(cos\theta + i\ sin\theta)^n = cosn\theta + i\ sinn\theta$

which is the De Moivre's theorem.

1.2. Function of a Complex Variables:

If $z = x + iy$ and $w = u + iv$ are two complex variables, and if for each value of z in a certain portion of the complex plane (called also as the domain R of the complex plane) there corresponds one or more values of w, then w is said to be a function of z and is written as

$$w = f(z) = f(x + iy) = u(x, y) + i\ v(x, y) \qquad (1)$$

where $u(x, y)$and $v(x, y)$ are real functions of the real variables x and y. Clearly for a given value of z, the values of x and y are known and thus, one or more values of w are determined by (1). If for each value of z in R, there is correspondingly only one value of w, then w is called a *single-valued function* of z. If there is more than one value of w corresponding to a given value of z, then w is called a *multiple-valued function or many- valued function*.

For example, $w = z^2, w = \frac{1}{z}, w = \frac{z}{z^4+1}$ are single valued function of z. The function $w = z^{1/2}$, $w = \arg(z)$ are examples of many valued functions. The first one has three values for each value of z (except for $z = 0$) and the second one assumes infinite set of real values for each value of z other than $z = 0$.

The complex quantities z and w can be represented on separate complex planes, called the z-plane and the w-plane respectively. The relation $w = f(z)$ establishes correspondence between the points (x, y) of the z-plane and the points (u, v) of the w-plane.

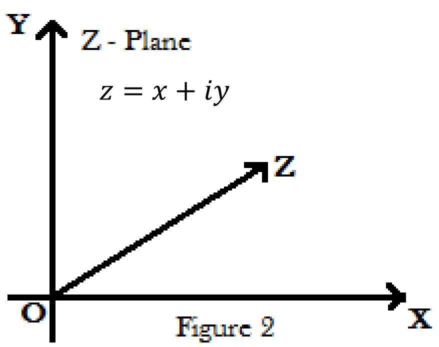


Figure 2

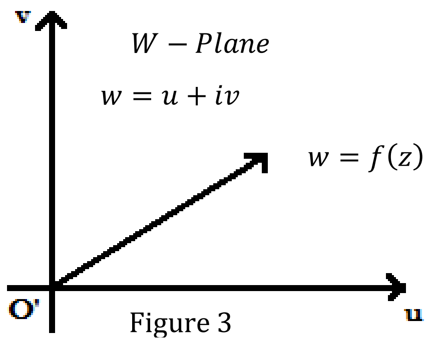


Figure 3

1.3. Limits: Let $w = f(z)$ denote some functional relationship connecting w with z.

Then $w = f(x+iy) = u(x,y) + i\,v(x,y)$ where u and v are real functions of x and y. As z approaches z_0, the limit of $f(z)$ is said to be w_0 if $f(z)$ can be kept arbitrarily close to w_0, by keeping z sufficiently close to, but different from z_0.

$$i.e., \lim_{z \to z_0} w = \lim_{z \to z_0} f(z) = w_0$$

Now let $z_0 = x_0 + iy_0$

when z approaches z_0, it means that $x \to x_0$ and $y \to y_0$.

Hence $\lim_{z \to z_0} f(z) = \lim_{z \to z_0} (u + i\,v) = \lim_{\substack{x \to x_0 \\ y \to y_0}} (u + i\,v) = u_0 + i\,v_0$

Hence $\lim_{\substack{x \to x_0 \\ y \to y_0}} u(x,y) = u_0$ and $\lim_{\substack{x \to x_0 \\ y \to y_0}} v(x,y) = v_0$.

Note: In the above, when we say that $z \to z_0$, it means that $x \to x_0$ and $y \to y_0$ in any order, by any path as shown in figure 4.

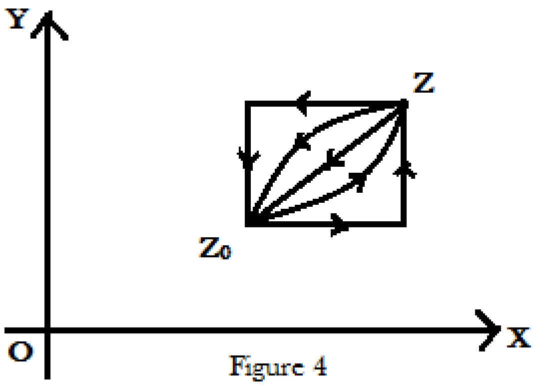


Figure 4

1.4. Continuity: The idea of continuity is closely connected with the concept of a limit. A single-valued function $w = f(z)$ is said to be continuous at a point $z = z_0$ provided each of the following conditions is satisfied:

(i) $f(z_0)$ exists

(ii) $\lim_{z \to z_0} f(z)$ exists, and

(iii) $\lim_{z \to z_0} f(z) = f(z_0)$

Remarks:

1. If $f(z)$ is continuous at every point of a region R, it is said to be continuous throughout R.

2. $w = f(z) = u(x,y) + i\,v(x,y)$. If $f(z)$is continuous at $z = z_0$, then its real and imaginary parts, i.e., u and v will be continuous functions at $z = z_0$, i.e., at $x = x_0$ and $y =$

y_0. Conversely, if u and v are continuous functions at $z = z_0$, then $f(z)$ will be continuous at $z = z_0$.

3. The sums, differences and products of continuous functions are also continuous are also continuous. The quotient of two continuous functions is continuous except for those values of z for which the denominator vanishes.

1.5. Continuity of a Function of Two Real Variables:

$$w = f(z) = f(x + iy)$$

is a function of the two variables x and y. Hence, to discuss the continuity of $f(z)$, we shall have to deal with the continuity of a function of two independent variables x and y.

Definition: a function $f(x,\ y)$ of two real independent variables x and y is said to be continuous at a point $(x_0,\ y_0)$ if,

(i) $f(x_0,\ y_0)$, the value of $f(x,\ y)$ at $(x_0,\ y_0)$ is finite, and

(ii) $\lim\limits_{\substack{x\to x_0 \\ y\to y_0}} f(x,y) = f(x_0,\ y_0)$ in whatever way $x \to x_0$ and $y \to y_0$

To illustrate the idea of continuity of a function of two variables given in the following examples:

EX. 1. Show that $f(x,y) = \frac{2xy}{x^2 + y^2}$ is discontinuous at origin, given that $f(0,0) = 0$.

Solution: Given $f(x,y) = \frac{2xy}{x^2 + y^2}$

If $y \to 0$ first and then $x \to 0$

$$\lim_{x\to 0}\lim_{y\to 0} f(x,y) = \lim_{x\to 0}\lim_{y\to 0}\frac{2xy}{x^2 + y^2} = \lim_{x\to 0}\frac{2x(0)}{x^2} = 0$$

If $x \to 0$ first and then $y \to 0$

$$\lim_{y\to 0}\lim_{x\to 0} f(x,y) = \lim_{y\to 0}\lim_{x\to 0}\frac{2xy}{x^2 + y^2} = \lim_{y\to 0}\frac{2y(0)}{y^2} = 0$$

Let x and y both tend to zero simultaneously along the path $y = mx$.

Then, $\lim\limits_{\substack{y=mx \\ x\to 0}} f(x,y) = \lim\limits_{\substack{y=mx \\ x\to 0}}\frac{2xy}{x^2 + y^2} = \lim\limits_{x\to 0}\frac{2x.mx}{x^2 + m^2x^2} = \frac{2m}{1 + m^2}$

This limit changes its value for different values of m.

when $m = 1, \frac{2m}{1 + m^2} = 1$ and for $m = 2, \frac{2m}{1 + m^2} = \frac{4}{5}$ and so on.

Hence $\lim\limits_{y\to 0}\frac{2xy}{x^2 + y^2} \neq 0$, when $x \to 0$, $y \to 0$ in any manner. So the function is not continuous at the origin.

EX. 2. Discuss the continuity of $f(x,y) = \frac{2xy^2}{x^2+y^4}$ at the origin, given that $f(0,0) = 0$.

Solution: Given $f(x,y) = \frac{2xy^2}{x^2+y^4}$

If $y \to 0$ first and then $x \to 0$

$$\lim_{x\to 0}\lim_{y\to 0} f(x,y) = \lim_{x\to 0}\lim_{y\to 0} \frac{2xy^2}{x^2 + y^4} = \lim_{x\to 0} \frac{2x(0)}{x^2} = 0$$

If $x \to 0$ first and then $y \to 0$

$$\lim_{y\to 0}\lim_{x\to 0} f(x,y) = \lim_{y\to 0}\lim_{x\to 0} \frac{2xy^2}{x^2 + y^4} = \lim_{y\to 0} \frac{2y^2(0)}{y^4} = 0$$

Let x and y both tend to zero simultaneously along the path $y^2 = x$.

$$\text{Then, } \lim_{\substack{x=y^2\\ y\to 0}} f(x,y) = \lim_{\substack{x=y^2\\ y\to 0}} \frac{2xy^2}{x^2 + y^4} = \lim_{y\to 0} \frac{2\,y^4}{2\,y^4} = 1 \neq 0$$

Hence, the function is discontinuous at the origin.

1.6. Derivative of a Function of a Complex Variable: For a real function of a single real variable say, $y = f(x)$, the derivative of y with Respect to x is defined as

$$\frac{dy}{dx} = \lim_{\Delta x\to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

Hence Δx can approach zero in only one way.

Let $w = f(z)$ be a single-valued function of z. Then, the derivative of w is defined to be

$$\frac{dw}{dz} = f'(z) = \lim_{\Delta z\to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

provided the above limit exists and is the same, in whatever manner Δz approaches zero.

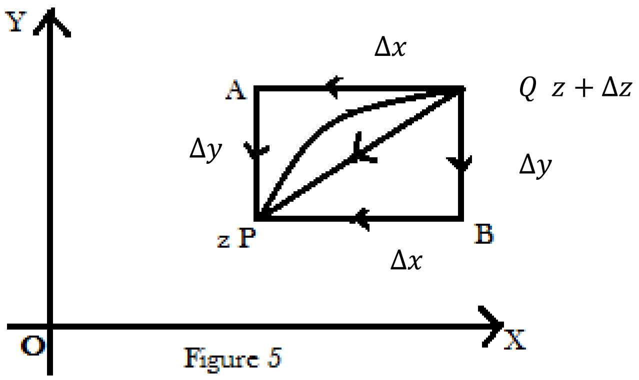


Figure 5

We can show by a figure that Δz can approach zero in several ways. P is the point in the z-plane corresponding to $z = x + i\,y$. Q is the point $z + \Delta z$. $\Delta z = \Delta x + i\,\Delta y$, where

$\Delta x, \Delta y$ are small increments in x and y respectively. As $\Delta z \to 0,$ $i.e., \Delta x, \Delta y also \to 0$ and the point Q approaches to P. Now Q can approach P along the rectilinear path QAP on which first Δx and then Δy approach zero or Q may approach P along the rectilinear path QBP on which first Δy and then Δx approach zero. More generally, Q can approach P along infinitely many paths, $i.e.,$ Δz approaches zero in several ways.

Hence, in the definition of $f'(z)$, the derivative of $f(z)$, it is necessary that the limit of the difference quotient

$$i.e., \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}$$

should be the same, no matter how Δz approaches zero. When this limit is unique, the function is said to be differentiable. This severe restriction narrows down greatly the class of functions of a complex variable that possess derivatives.

Thus we find that $\frac{dw}{dz}$ depends not only upon z but also upon the manner in which Δz approaches zero. To illustrate this, consider the simple case,

$$w = f(z) = x - i\,y$$

Then

$$\frac{f(z + \Delta z) - f(z)}{\Delta z} = \frac{[(x + \Delta x) - i\,(y + \Delta y)] - (x - i\,y)}{\Delta x + i\,\Delta y}$$

$$= \frac{\Delta x - i\,\Delta y}{\Delta x + i\,\Delta y}$$

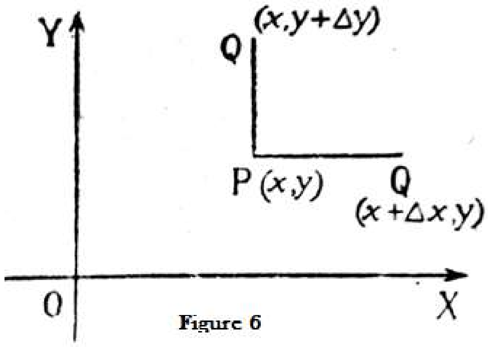


Figure 6

Now, let $\Delta z \to 0$ is such a way that first Δy and then Δx approach zero, i.e., Q approaches P along the horizontal line. Then

$$\lim_{\Delta z \to 0} \frac{\Delta x - i\,\Delta y}{\Delta x + i\,\Delta y} = \lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x} = 1$$

But, suppose Q approaches P along the vertical line so that first Δx and then Δy approach zero. Then

$$\lim_{\Delta z \to 0} \frac{\Delta x - i\,\Delta y}{\Delta x + i\,\Delta y} = \lim_{\Delta y \to 0} \frac{-i\,\Delta y}{i\,\Delta y} = -1$$

For other paths of approach of Q towards P, we can get as many distinct values of the above limit as we please. We therefore say that $f(z) = x - i\,y$ possesses no derivative.

Definition: If a single-valued function $w = f(z)$ possesses a derivative at $z = z_0$ and at every point in some neighbourhood of z_0, then $f(z)$ is said to be *analytic* at z_0 and z_0 is called a *regular point* of the function. If $f(z)$ is analytic at every point of a region R, then we say that $f(z)$is analytic in R. A point at which an analytic function ceases to have a derivative is called a singular point. An analytic function is also referred to as *regular* or *holomorphic*.

1.7. Conditions under which $w = f(z)$is analytic:

Let $w = f(z)$ be an analytic function of a complex variable in a region R. Then $f'(z)$ exists at every point in R. Let us now find the conditions for the existence of the derivative of $f(z)$ at a point z.

Let $z = x + i\,y$ and $w = f(z) = f(x + i\,y) = u(x, y) + i\,v(x, y)$

where u and v are functions of x and y. Let Δx and Δy be the increments in x and y respectively and let Δz be the corresponding increment in z

Then $z + \Delta z = (x + \Delta x) + i(y + \Delta y)$

Hence $\Delta z = \Delta x + i\,\Delta y$

Also $f(z + \Delta z) = u(x + \Delta x, y + \Delta y) + i\,v(x + \Delta x, y + \Delta y)$

Hence $\frac{f(z+\Delta z) - f(z)}{\Delta z} = \frac{[u(x+\Delta x, y+\Delta y) + i\,v(x+\Delta x, y+\Delta y)] - [u(x,y) + i\,v(x,y)]}{\Delta x + i\,\Delta y}$

As $\Delta z \to 0$, we have $\Delta x \to 0$ and $\Delta y \to 0$.

Hence by definition,

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}$$

$$f'(z) = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{[u(x + \Delta x, y + \Delta y) + i\,v(x + \Delta x, y + \Delta y)] - [u(x, y) + i\,v(x, y)]}{\Delta x + i\,\Delta y} \qquad (1)$$

If $f(z)$ is analytic, $f'(z)$ must have a unique value, in whatever manner $\Delta z \to 0$. Now let $\Delta z \to 0$ in such a way that first Δy and then $\Delta x \to 0$. Then from (1),

$$f'(z) = \lim_{\Delta x \to 0} \frac{[u(x + \Delta x, y) + i\,v(x + \Delta x, y)] - [u(x, y) + i\,v(x, y)]}{\Delta x}$$

$$i.e., f'(z) = \lim_{\Delta x \to 0} \frac{[u(x + \Delta x, y) - u(x, y)] + i\,[v(x + \Delta x, y) - v(x, y)]}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{u(x+\Delta x, y) - u(x,y)}{\Delta x} + i \lim_{\Delta x \to 0} \frac{v(x+\Delta x, y) - v(x,y)}{\Delta x}$$

$$= \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \qquad (2)$$

(by definition of partial derivatives)

Since $f'(z)$ is to be unique, it is necessary that the partial derivatives $\frac{\partial u}{\partial x}$ and $\frac{\partial v}{\partial x}$ must exists at the point (x, y).

Secondly, let $\Delta z \to 0$ such that $\Delta x \to 0$ first and then $\Delta y \to 0$. Then from (1)

$$f'(z) = \lim_{\Delta y \to 0} \frac{[u(x, y+\Delta y) + i\, v(x, y+\Delta y)] - [u(x,y) + i\, v(x,y)]}{i\, \Delta y}$$

$$i.e., f'(z) = \lim_{\Delta y \to 0} \frac{[u(x, y+\Delta y) - u(x,y)] + i\, [v(x, y+\Delta y) - v(x,y)]}{i\, \Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{u(x, y+\Delta y) - u(x,y)}{i\, \Delta y} + \lim_{\Delta y \to 0} \frac{v(x, y+\Delta y) - v(x,y)}{\Delta y}$$

$$= \frac{1}{i} \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} = \frac{\partial v}{\partial y} - i\, \frac{\partial u}{\partial y} \qquad (3)$$

Hence $\frac{\partial u}{\partial y}$ and $\frac{\partial v}{\partial y}$ must exist at (x, y).

Now, if the derivative $f'(z)$ exists, it is necessary that the two expressions (2) and (3) which we have derived for it must be the same. Hence equating these expressions, we have

$$\frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\, \frac{\partial u}{\partial y}$$

Equating real and imaginary parts, we get

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad (4)$$

$$\text{and } \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \qquad (5)$$

$$i.e., u_x = v_y \text{ and } v_x = -u_y$$

The equations (4) and (5) are called ***Cauchy-Riemann differential equations***.

Note: The Cauchy-Riemann equations are only the necessary conditions for the function $f(z) = u + i\, v$ to be differentiable i.e., if the function is differentiable, then it must satisfy these equations. But the converse is not necessarily true. A function may satisfy these equations at a point and yet it may not be differentiable at that point.

Hence the conditions expressed by Cauchy-Riemann equations (C-R equations) are only *necessary but not sufficient* for a function to be analytic.

1.8. Sufficient Conditions for $f(z)$ to be Analytic: We shall now prove the following theorem

The single valued continuous function $w = f(z) = u + i\,v$ analytic in a region R, if the four partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}$ *and* $\frac{\partial v}{\partial y}$ exist, are continuous and satisfy the ***Cauchy-Riemann equations*** at each point in$R$.

Proof: Let $w = f(z) = u(x,y) + i\,v(x,y)$

It is now given that

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \; and \; \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \qquad (1)$$

Also these partial derivatives are continuous.

Then $\Delta u = u(x+\Delta x, y+\Delta y) - u(x,y)$

$$= [u(x+\Delta x, y+\Delta y) - u(x+\Delta x, y)] + [u(x+\Delta x, y) - u(x,y)]$$

$$= \Delta y.\frac{\partial}{\partial y}u(x+\Delta x, y+\theta_1.\Delta y) + \Delta x.\frac{\partial}{\partial x}u(x+\theta_2.\Delta x, y)$$

Using the first Mean Value Theorem, θ_1 and θ_2 being both positive and less than 1.

Now, at the point (x,y) the derivatives $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ are continuous.

Hence the above expression Δu may be written as

$$\Delta u = \Delta x.\left[\frac{\partial u}{\partial x} + \lambda_1\right] + \Delta y.\left[\frac{\partial u}{\partial y} + \lambda_2\right] \qquad (2)$$

where λ_1 and λ_2 both tend to zero as $|\Delta z| \to 0$

Similarly, using the result that the derivatives $\frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$ are continuous, we get

$$\Delta v = \Delta x.\left[\frac{\partial v}{\partial x} + \mu_1\right] + \Delta y.\left[\frac{\partial v}{\partial y} + \mu_2\right] \qquad (3)$$

where μ_1 and μ_2 both tend to zero as $|\Delta z| \to 0$

Now $\Delta w = \Delta u + i\,\Delta v$

$$= \left\{\Delta x.\left[\frac{\partial u}{\partial x} + \lambda_1\right] + \Delta y.\left[\frac{\partial u}{\partial y} + \lambda_2\right]\right\} + i\left\{\Delta x.\left[\frac{\partial v}{\partial x} + \mu_1\right] + \Delta y.\left[\frac{\partial v}{\partial y} + \mu_2\right]\right\}$$

$$= \Delta x\left(\frac{\partial u}{\partial x} + i\,\frac{\partial v}{\partial x}\right) + \Delta y\,\left(\frac{\partial u}{\partial y} + i\,\frac{\partial v}{\partial y}\right) + \varepsilon_1\,\Delta x + \varepsilon_2\,\Delta y \qquad (4)$$

where $\varepsilon_1 = \lambda_1 + i\,\mu_1$ and $\varepsilon_2 = \lambda_2 + i\,\mu_2$ and $\varepsilon_{1,}\varepsilon_2 \to 0$ as $|\Delta z| \to 0$.

In (4), apply the conditions (1) *i.e.*, put

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y} \text{ and } \frac{\partial v}{\partial x}=-\frac{\partial u}{\partial y}$$

Then $\Delta w=\Delta x\left(\frac{\partial u}{\partial x}+i\ \frac{\partial v}{\partial x}\right)+\Delta y\ \left(-\frac{\partial v}{\partial x}+i\ \frac{\partial u}{\partial x}\right)+\varepsilon_1\ \Delta x+\varepsilon_2\ \Delta y$

$=(\Delta x+i\ \Delta y)\frac{\partial u}{\partial x}+i\ (\Delta x+i\ \Delta y)\frac{\partial v}{\partial x}+\varepsilon_1\ \Delta x+\varepsilon_2\ \Delta y$

$=(\Delta x+i\ \Delta y)\left[\frac{\partial u}{\partial x}+i\ \frac{\partial v}{\partial x}\right]+\varepsilon_1\ \Delta x+\varepsilon_2\ \Delta y$

Hence $$\frac{\Delta w}{\Delta z}=\frac{\partial u}{\partial x}+i\ \frac{\partial v}{\partial x}+\varepsilon_1\frac{\Delta x}{\Delta z}+\varepsilon_2\frac{\Delta y}{\Delta z} \qquad (5)$$

Now $|\Delta x|\le|\Delta z|$ and $|\Delta y|\le|\Delta z|$

and so $\left|\frac{\Delta x}{\Delta z}\right|\le 1$ and $\left|\frac{\Delta y}{\Delta z}\right|\le 1.$

Also $\varepsilon_{1,}\varepsilon_2\to 0$ as $|\Delta z|\to 0$

So proceeding to the limit as $\Delta z\to 0$, (5) gives

$$\frac{dw}{dz}=\frac{\partial u}{\partial x}+i\ \frac{\partial v}{\partial x}$$

$i.e., f'(z)$ exists and is equal to $\frac{\partial u}{\partial x}+i\ \frac{\partial v}{\partial x}$

We shall put the above discussion in 4.7 and 4.8 relating to differentiability in the form of a theorem as follows.

If u and v are real single-valued functions of x and y which, with their four first order partial derivatives$\left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}\right.$ and $\left.\frac{\partial v}{\partial y}\right)$, are continuous throughout a region R, then the ***Cauchy-Riemann equations***

$$u_x=v_y \ and\ v_x=-u_y$$

are both necessary and sufficient condition, so that $f(z)=u+i\ v$ may be analytic. The derivative of $f(z)$ is then given by either of the expressions

$$f'(z)=\frac{\partial u}{\partial x}+i\ \frac{\partial v}{\partial x} \text{ or } f'(z)=\frac{\partial v}{\partial y}-i\ \frac{\partial u}{\partial y}$$

1.9. Derive the *Cauchy-Riemann equations* if $f(z)$ is expressed in polar coordinates.

Solution: Let $f(z)=u(r,\theta)+i\ v(r,\theta)$ in polar coordinates.

$z=x+i\ y=r(cos\theta+i\ sin\theta)=r\ e^{i\theta}.$

Let $\Delta r\ and\ \Delta\theta$ be the increments in $r\ and\ \theta$ respectively and let Δz be the corresponding increment in z.

$$\Delta z=\Delta\left(r\ e^{i\theta}\right)$$

$$f(z+\Delta z) = u(r+\Delta r, \theta+\Delta\theta) + i\, v(r+\Delta r, \theta+\Delta\theta)$$

$$f(z+\Delta z) - f(z) = [u(r+\Delta r, \theta+\Delta\theta) + i\, v(r+\Delta r, \theta+\Delta\theta)] - [u(r,\theta) + i\, v(r,\theta)]$$

Hence $\quad \dfrac{f(z+\Delta z) - f(z)}{\Delta z}$

$$= \frac{[u(r+\Delta r, \theta+\Delta\theta) + i\, v(r+\Delta r, \theta+\Delta\theta)] - [u(r,\theta) + i\, v(r,\theta)]}{\Delta z}$$

$$= \frac{[u(r+\Delta r, \theta+\Delta\theta) + i\, v(r+\Delta r, \theta+\Delta\theta)] - [u(r,\theta) + i\, v(r,\theta)]}{\Delta(r\, e^{i\theta})}$$

By definition, $\quad f'(z) = \lim\limits_{\Delta z \to 0} \dfrac{f(z+\Delta z) - f(z)}{\Delta z}$

$$= \lim_{\Delta z \to 0} \frac{[u(r+\Delta r, \theta+\Delta\theta) + i\, v(r+\Delta r, \theta+\Delta\theta)] - [u(r,\theta) + i\, v(r,\theta)]}{\Delta(r\, e^{i\theta})} \qquad (1)$$

If $f(z)$ is analytic, $f'(z)$ must have a unique value in whatever manner $\Delta z \to 0$.

First let $\Delta z \to 0$ along a radius vector through the origin.

i.e., keep θ constant.

Then $\Delta z = \Delta(r\, e^{i\theta}) = e^{i\theta}\Delta r$.

As $\Delta z \to 0$, $\Delta r \to 0$. So (1) gives

$$f'(z) = \lim_{\Delta r \to 0} \frac{[u(r+\Delta r, \theta) + i\, v(r+\Delta r, \theta)] - [u(r,\theta) + i\, v(r,\theta)]}{e^{i\theta}\Delta r}$$

$$= e^{-i\theta} \lim_{\Delta r \to 0} \left[\frac{u(r+\Delta r, \theta) - u(r,\theta)}{\Delta r} + i\, \frac{v(r+\Delta r, \theta) - v(r,\theta)}{\Delta r}\right]$$

$$= e^{-i\theta} \left[\lim_{\Delta r \to 0} \frac{u(r+\Delta r, \theta) - u(r,\theta)}{\Delta r} + i \lim_{\Delta r \to 0} \frac{v(r+\Delta r, \theta) - v(r,\theta)}{\Delta r}\right]$$

$$= e^{-i\theta} \left(\frac{\partial u}{\partial r} + i\, \frac{\partial v}{\partial r}\right) \qquad (2)$$

Secondly, keep r constant.

Then $\Delta z = \Delta(r\, e^{i\theta}) = ire^{i\theta}\Delta\theta$

As $\Delta z \to 0$, $\Delta\theta \to 0$. So (1) gives

$$f'(z) = \lim_{\Delta\theta \to 0} \frac{[u(r, \theta+\Delta\theta) + i\, v(r, \theta+\Delta\theta)] - [u(r,\theta) + i\, v(r,\theta)]}{ire^{i\theta}\Delta\theta}$$

$$= \frac{1}{re^{i\theta}} \lim_{\Delta\theta \to 0} \frac{[u(r, \theta+\Delta\theta) + i\, v(r, \theta+\Delta\theta)] - [u(r,\theta) + i\, v(r,\theta)]}{i\Delta\theta}$$

$$= \frac{1}{re^{i\theta}} \lim_{\Delta\theta \to 0} \frac{[u(r, \theta+\Delta\theta) - u(r,\theta)] + i[\, v(r, \theta+\Delta\theta) - \, v(r,\theta)]}{i\Delta\theta}$$

$$= \frac{1}{re^{i\theta}}\left[-i \lim_{\Delta\theta\to 0} \frac{u(r,\theta+\Delta\theta)-u(r,\theta)}{\Delta\theta} + \lim_{\Delta\theta\to 0} \frac{v(r,\theta+\Delta\theta)-v(r,\theta)}{\Delta\theta}\right]$$

$$= \frac{1}{r} e^{-i\theta}\left(-i \frac{\partial u}{\partial\theta} + \frac{\partial v}{\partial\theta}\right) \qquad (3)$$

Since $f(z)$ is analytic, $f'(z)$ must have a unique value in whatever manner $\Delta z \to 0$. Then From (2) and (3), we get

$$\frac{\partial u}{\partial r} + i\,\frac{\partial v}{\partial r} = \frac{1}{r}\left(-i\,\frac{\partial u}{\partial\theta} + \frac{\partial v}{\partial\theta}\right)$$

Equating on both sides real and imaginary parts, we get

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial\theta} \qquad (4)$$

$$\text{and } \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial\theta} \qquad (5)$$

These equations are the ***Cauchy-Riemann equations*** if $f(z)$ is expressed in polar coordinates.

Note: Differentiating (4) partially with respect to r, we get

$$\frac{\partial^2 u}{\partial r^2} = -\frac{1}{r^2}\frac{\partial v}{\partial\theta} + \frac{1}{r}\frac{\partial^2 v}{\partial r\,\partial\theta} \qquad (6)$$

Differentiating (5) partially with respect to θ, we get

$$\frac{\partial^2 u}{\partial\theta^2} = -r\,\frac{\partial^2 v}{\partial\theta\,\partial r} \qquad (7)$$

Thus using (4), (6) and (7), we get

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2} = 0 \quad \left(\text{since} \frac{\partial^2 v}{\partial r\,\partial\theta} = \frac{\partial^2 v}{\partial\theta\,\partial r}\right)$$

EX. 3. Show that $w = f(z) = \bar{z} = x - i\,y$ is not analytic anywhere in the complex plane.

Solution: Let $w = u + i\,v = x - i\,y$.

Here $u = x$ and $v = -y$

Then $\frac{\partial u}{\partial x} = 1, \frac{\partial u}{\partial y} = 0, \frac{\partial v}{\partial x} = 0 \; and \; \frac{\partial v}{\partial y} = -1$

Hence $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ but $\frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y}$

The second of the Cauchy-Riemann equations is satisfied everywhere, but not so the first. So $w = \bar{z}$ is not analytic anywhere in the complex plane.

EX. 4. Show that $w = f(z) = z = x + i\,y$ is analytic anywhere in the complex plane.

Solution: Let $w = u + i\,v = x + i\,y$.

Here $u = x$ and $v = y$

Then $\frac{\partial u}{\partial x} = 1, \frac{\partial u}{\partial y} = 0, \frac{\partial v}{\partial x} = 0 \; and \; \frac{\partial v}{\partial y} = 1$

Hence $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ and $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ at all points in the complex plane. The C-R equations are identically satisfied. Further these four partial derivatives are continuous.

Hence $w = f(z) = z$is analytic anywhere in the complex plane.

EX. 5. Show that $w = f(z) = e^z$ is analytic everywhere in the complex plane and find $f'(z)$.

Solution: Let $w = f(z) = e^z = e^{x+i\,y} = e^x(cosy + i\,siny) = u + iv$

Here $u = e^x\,cosy$ and $v = e^x\,siny$.

Then $\frac{\partial u}{\partial x} = e^x\,cosy, \frac{\partial u}{\partial y} = -e^x\,siny, \frac{\partial v}{\partial x} = e^x\,siny \; and \; \frac{\partial v}{\partial y} = e^x\,cosy$

Clearly $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ but $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$

The C-R equations are identically satisfied. Also the partial derivatives are continuous.

Hence $f'(z)$ exists at all points of the z plane i.e., $f(z)$ is analytic everywhere.

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$= e^x\,cosy + i\,e^x\,siny$$

$$= e^x(cosy + i\,siny)$$

$$= e^x e^{iy} = e^{x+iy} = e^z$$

EX. 6. Show that $w = f(z) = z\bar{z}$ is differentiable but not analytic at the point $z = 0$.

Solution: Let $w = f(z) = z\bar{z} = (x + i\,y)(x - i\,y) = x^2 + y^2 = u + i\,v$

Here $u = x^2 + y^2$ and $v = 0$

Then $\frac{\partial u}{\partial x} = 2x, \frac{\partial u}{\partial y} = 2y, \frac{\partial v}{\partial x} = 0 \; and \; \frac{\partial v}{\partial y} = 0$

Now $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ only when $x = 0$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ only when $y = 0$.

Hence the C-R equations are satisfied only when both x and y are zero. i.e., they are satisfied only at the origin. Hence $f(z)$ has a derivative at $z = 0$.

But the C-R equations are not satisfied for $z \neq 0$. Hence there is no neighbourhood about $z = 0$ in which the function is differentiable. Hence it is not analytic at $z = 0$.

EX. 7. Test whether $w = f(z) = z^3$ is analytic or not.

Solution: Given $w = f(z) = z^3 = (x + iy)^3 = x^3 + 3ix^2y + 3i^2xy^2 + i^3y^3$

$$= (x^3 - 3xy^2) + i(3x^2y - y^3) = u + iv$$

Here $u = x^3 - 3xy^2$ and $v = 3x^2y - y^3$

Then $\frac{\partial u}{\partial x} = 3x^2 - 3y^2, \frac{\partial u}{\partial y} = -6xy, \frac{\partial v}{\partial x} = 6xy \; and \; \frac{\partial v}{\partial y} = 3x^2 - 3y^2$

Clearly, $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ for all values of x and y.

The C-R equations are identically satisfied. Further these four partial derivatives are continuous.

Hence $w = f(z) = z^3$is analytic.

EX. 8. Verify whether $w = sinx \; cos \; hy + i \; cosx \; sin \; hy$ is analytic or not.

Solution: Given $w = sinx \; cos \; hy + i \; cosx \; sin \; hy = u + iv$

Here $u = sinx \; cos \; hy$ and $v = cosx \; sin \; hy$

Then $\frac{\partial u}{\partial x} = cosx \; coshy, \frac{\partial u}{\partial y} = sinx \; sin \; hy, \frac{\partial v}{\partial x} = -sinx \; sin \; hy \; and \; \frac{\partial v}{\partial y} = cosx \; coshy$

Clearly, $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ for all values of x and y.

The C-R equations are identically satisfied. Further these four partial derivatives are continuous.

Hence $w = sinx \; cos \; hy + i \; cosx \; sin \; hy$ is analytic.

EX. 9. State whether $sin(x - iy)$ is analytic or not.

Solution: Given $w = sin(x - iy) = sinx \cos \; iy - cosx \sin iy$

$$= sinx \; cos \; hy - i \; cosx \; sin \; hy = u + iv$$

Here $u = sinx \; cos \; hy$ and $v = -cosx \; sin \; hy$

Then $\frac{\partial u}{\partial x} = cosx \; coshy, \frac{\partial u}{\partial y} = sinx \; sin \; hy, \frac{\partial v}{\partial x} = sinx \; sin \; hy \; and \; \frac{\partial v}{\partial y}$

$$= -cosx \; coshy$$

Clearly $\frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} \neq -\frac{\partial v}{\partial x}$.

The C-R equations are not satisfied.

Hence $w = sin(x - iy)$ is not analytic.

EX. 10. If $f(z) = u + iv$ is analytic, then $v + iu$is analytic or not.

Solution: Given $f(z) = u + iv$ is analytic

Then u and v are satisfy C-R equations

$$i.e., \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad (1)$$

$$\text{and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \qquad (2)$$

If $v + iu$ is analytic, we must have

$$\frac{\partial v}{\partial x} = \frac{\partial u}{\partial y} \; and \; \frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x}$$

i.e., we must have $\frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y}$ and$\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}$.

But these are contrary to (1) and (2). So $v + iu$ is not analytic.

EX. 11. If $f(z) = u + iv$ is analytic function, find the condition under which $v + iu$ will be analytic.

Solution: Given $f(z) = u + iv$ is analytic

Then u and v are satisfy C-R equations

$$i.e., \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad (1)$$

$$\text{and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} (2)$$

If $v + iu$ is analytic, we must have

$$\frac{\partial v}{\partial x} = \frac{\partial u}{\partial y} \qquad (3)$$

$$and \; \frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x} \qquad (4)$$

Combining (1) and (4), we have

$$\frac{\partial u}{\partial x} = -\frac{\partial u}{\partial x} \;\; i.e., \frac{\partial u}{\partial x} = 0 \qquad (5)$$

Combining (2) and (3), we have

$$\frac{\partial u}{\partial y} = -\frac{\partial u}{\partial y} \; i.e., \frac{\partial u}{\partial y} = 0 \qquad (6)$$

From (5) and (6) it is clear that u is a constant, independent of x and y.

Similarly, we can prove that $\frac{\partial v}{\partial x} = 0 \; and \; \frac{\partial v}{\partial y} = 0$

$i.e., v$ is a constant, independent of x and y.

So the required conditions are u and v must be both constants.

EX. 12. (a) If $u + iv$ as well as $u - iv$ are analytic. What can you say about $u \; and \; v$.

(b) If both $f(z) and \; \bar{f}(z)$ are analytic functions, show that $f(z)$ is a constant.

Solution: (a) since $f(z) = u + iv$ is analytic

Then u and v are satisfy C-R equations

$$i.e., \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad (1)$$

$$\text{and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} (2)$$

Since $\bar{f}(z) = u - iv$ is also analytic.

Then u and v are satisfy C-R equations

$$i.e., \frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y} \qquad (3)$$

$$\text{and } \frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} \qquad (4)$$

Combining (1) and (3), we have

$$\frac{\partial u}{\partial x} = -\frac{\partial u}{\partial x} \;\; i.e., \frac{\partial u}{\partial x} = 0 \qquad (5)$$

Combining (2) and (4), we have

$$\frac{\partial u}{\partial y} = -\frac{\partial u}{\partial y} \; i.e., \frac{\partial u}{\partial y} = 0 \qquad (6)$$

From (5) and (6) it is clear that u is a constant, independent of x and y.

Similarly, we can prove that $\frac{\partial v}{\partial x} = 0$ and $\frac{\partial v}{\partial y} = 0$

$i.e., v$ is a constant, independent of x and y.

So the required conditions are u and v must be both constants.

(b) Let $f(z) = u + iv$. Then $\bar{f}(z) = u - iv$

It is given that $f(z) and \; \bar{f}(z)$ are analytic functions, so u and v must be both constants (refer (a)).

Therefore $f(z)$ is a constant.

EX. 13. If $u + iv$ is analytic, show that $v - iu$ and $-v + iu$ are also analytic.

Solution: We know that if $u + iv$ is analytic, $k(u + iv)$ is also analytic, where k is a constant.

Hence (i) taking $k = i$ we have $i(u + iv) = -v + iu$ is also analytic

(ii) taking $k = -i$ we have $-i(u + iv) = v - iu$ is also analytic

EX. 14. Show that $w = f(z) = z^n$ is analytic for positive integral values of n and find $f'(z)$.

Solution: Using polar coordinates, let $f(z) = u(r, \theta) + i\, v(r, \theta)$ and let $z = re^{i\theta}$.

Then $f(z) = u(r, \theta) + i\, v(r, \theta)$

$$= z^n = \left(re^{i\theta}\right)^n$$

$$= r^n(\cos\ n\theta + i \sin n\theta)$$

Here $u = r^n cos\, n\theta$ and $v = r^n sin\, n\theta$

Then $\dfrac{\partial u}{\partial r} = nr^{n-1} \cos n\theta\, , \dfrac{\partial u}{\partial \theta} = -nr^n \sin n\theta\, , \dfrac{\partial v}{\partial r} = nr^{n-1} sin\, n\theta$ and $\dfrac{\partial v}{\partial \theta}$

$$= nr^n cos\, n\theta$$

Clearly $\dfrac{\partial u}{\partial r} = \dfrac{1}{r}\dfrac{\partial v}{\partial \theta}$ and $\dfrac{\partial v}{\partial r} = -\dfrac{1}{r}\dfrac{\partial u}{\partial \theta}$.

Hence the C-R equations are satisfied. Further the partial derivatives are continuous.

So $f(z) = z^n$ is analytic.

Now $f'(z) = e^{-i\theta}\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right) = e^{-i\theta}(nr^{n-1} \cos n\theta + i\, nr^{n-1} sin\, n\theta)$

$$= e^{-i\theta}\, nr^{n-1}(\cos n\theta + i\, sin\, n\theta)$$

$$= e^{-i\theta}\, nr^{n-1}e^{in\theta}$$

$$= nr^{n-1}e^{i(n-1)\theta} = n\left(re^{i\theta}\right)^{n-1}$$

$$= nz^{n-1}$$

EX. 15. Show that $w = f(z) = logz$ is analytic everywhere in the complex plane except at the origin and that its derivative is $\frac{1}{z}$.

Solution: Using polar coordinates, let $f(z) = u(r, \theta) + i\, v(r, \theta)$ and let $z = re^{i\theta}$.

Then $f(z) = u(r, \theta) + i\, v(r, \theta) = logz$

$$= log\left(re^{i\theta}\right) = logr + i\theta$$

Here $u = logr$ and $v = \theta$

Then $\dfrac{\partial u}{\partial r} = \dfrac{1}{r}, \dfrac{\partial u}{\partial \theta} = 0, \dfrac{\partial v}{\partial r} = 0$ and $\dfrac{\partial v}{\partial \theta} = 1$

Clearly $\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$ and $\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$.

Hence the C-R equations are satisfied. Further the partial derivatives are continuous at all point except when $r = 0$, i.e., except at the origin.

So $f(z) = logz$ is analytic everywhere except at the origin.

Now $f'(z) = e^{-i\theta}\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right) = e^{-i\theta}\left(\frac{1}{r}\right) = \frac{1}{re^{i\theta}} = \frac{1}{z}$

EX. 16. Prove that the function $f(z)$ where

$$f(z) = \frac{x^3(1+i) - y^3(1-i)}{x^2+y^2}, when\ z \neq 0, f(0) = 0$$

is continuous at $z = 0$. Prove also that the C-R equations are satisfied by $f(z)$ at $z = 0$ and yet $f'(z)$ does not exist at $z = 0$.

Solution: Given that $f(z) = \frac{x^3(1+i) - y^3(1-i)}{x^2+y^2}, when\ z \neq 0$

$$\lim_{z\to 0} f(z) = \lim_{\substack{y\to 0\\ x\to 0}} \frac{x^3(1+i) - y^3(1-i)}{x^2+y^2}$$

$$= \lim_{x\to 0} \frac{x^3(1+i)}{x^2} = 0$$

$$\lim_{z\to 0} f(z) = \lim_{\substack{x\to 0\\ y\to 0}} \frac{x^3(1+i) - y^3(1-i)}{x^2+y^2}$$

$$= \lim_{y\to 0} \frac{-y^3(1-i)}{y^2} = 0$$

Also $f(0) = 0$ be given.

Hence

$$\lim_{z\to 0} f(z) = f(0)$$

When $x \to 0$ first and then $y \to 0$ and also When $y \to 0$ first and then $x \to 0$.

Let $x\ and\ y$ both tend to zero simultaneously along the path $y = mx^n$.

For $n = 1$, this is a straight line and for $n = 2, 3, \ldots$, we will get different curves passing through the points (x, y) and the origin. Then

$$\lim_{z\to 0} f(z) = \lim_{\substack{y=mx^n\\ x\to 0}} \frac{x^3(1+i) - y^3(1-i)}{x^2+y^2}$$

$$= \lim_{x\to 0} \frac{x^3(1+i) - (mx^n)^3(1-i)}{x^2+(mx^n)^2}$$

$$= \lim_{x \to 0} \frac{x^3[1 + i - m^3 x^{3n-3}(1 - i)]}{x^2[1 + m^2 x^{2n-2}]}$$

$$= \lim_{x \to 0} \frac{x[1 + i - m^3 x^{3n-3}(1 - i)]}{1 + m^2 x^{2n-2}}$$

$$= \lim_{x \to 0} \frac{x[1 + i - m^3 (x^{n-1})^3(1 - i)]}{1 + m^2 (x^{n-1})^2} = 0$$

(because when $n > 1, n - 1$ is positive and $\lim_{x \to 0} x^{n-1} = 0$)

When $n = 1$ the above limit

$$= \lim_{x \to 0} \frac{x[1 + i - m^3(1 - i)]}{1 + m^2} = 0$$

Hence $\lim_{z \to 0} f(z) = f(0)$ in whatever manner $z \to 0$.

Therefore $f(z)$ is continuous at the origin.

Now $f(z) = \dfrac{x^3 - y^3}{x^2 + y^2} + i \; \dfrac{x^3 + y^3}{x^2 + y^2} = u(x, y) + i\; v(x, y)$

Here $u(x, y) = \dfrac{x^3 - y^3}{x^2 + y^2}$ and $v(x, y) = \dfrac{x^3 + y^3}{x^2 + y^2}$

Since $f(0) = 0, u(0, 0) = 0 \; and \; v(0, 0) = 0.$

Now at origin

$$\frac{\partial u}{\partial x} = \lim_{x \to 0} \frac{u(x, 0) - u(0, 0)}{x}$$

$$= \lim_{x \to 0} \frac{x^3}{x^3} = 1$$

$$\frac{\partial u}{\partial y} = \lim_{y \to 0} \frac{u(0, y) - u(0, 0)}{y}$$

$$= \lim_{y \to 0} \frac{-y^3}{y^3} = -1$$

$$\frac{\partial v}{\partial x} = \lim_{x \to 0} \frac{v(x, 0) - v(0, 0)}{x}$$

$$= \lim_{x \to 0} \frac{x^3}{x^3} = 1$$

$$\frac{\partial v}{\partial y} = \lim_{y \to 0} \frac{v(0, y) - v(0, 0)}{y}$$

$$= \lim_{y \to 0} \frac{y^3}{y^3} = 1$$

Hence at origin,

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}\ and\ \frac{\partial v}{\partial x}=-\frac{\partial u}{\partial y}$$

So the C-R equations are satisfied at the origin.

Now, by the definition

$$f'(0)=\lim_{z\to 0}\frac{f(z)-f(0)}{z}=\lim_{z\to 0}\frac{f(z)}{z}$$

$$=\lim_{z\to 0}\frac{x^3(1+i)-y^3(1-i)}{(x^2+y^2)\,(x+i\,y)}$$

Let $y\to 0$ first and then $x\to 0$.

$$f'(0)=\lim_{\substack{y\to 0\\ x\to 0}}\frac{x^3(1+i)-y^3(1-i)}{(x^2+y^2)\,(x+i\,y)}$$

$$=\lim_{x\to 0}\frac{x^3(1+i)}{x^3}=1+i$$

If $x\to 0$ first and then $y\to 0$.

$$f'(0)=\lim_{\substack{x\to 0\\ y\to 0}}\frac{x^3(1+i)-y^3(1-i)}{(x^2+y^2)\,(x+i\,y)}$$

$$=\lim_{y\to 0}\frac{-y^3(1-i)}{i\,y^3}=\frac{-(1-i)}{i}=i+1$$

Generally when $z\to 0$ along the path $y=mx$,

$$f'(0)=\lim_{\substack{y=mx\\ x\to 0}}\frac{x^3(1+i)-y^3(1-i)}{(x^2+y^2)\,(x+i\,y)}$$

$$=\lim_{x\to 0}\frac{x^3(1+i)-(mx)^3(1-i)}{(x^2+m^2x^2)\,(x+i\,mx)}$$

$$=\frac{(1+i)-m^3(1-i)}{(1+m^2)\,(1+i\,m)}$$

This assumes different values, as m varies, $f'(z)$ has no unique value at origin, i.e., $f(z)$ is not differentiable at that point.

Hence we find that even at a point, if $f(z)$ is continuous and satisfies the C-R equations, the function need not be differentiable.

EX. 17. Show that the function $f(z)=\sqrt{|xy|}$ is not regular at the origin, although C-R equations are satisfied.

Solution: Let $f(z)=\sqrt{|xy|}=u+iv$

Here $u=\sqrt{|xy|}$ and $v=0$

Now at origin

$$\frac{\partial u}{\partial x} = \lim_{x\to 0} \frac{u(x,0) - u(0,0)}{x}$$

$$= \lim_{x\to 0} \frac{0-0}{x} = 0$$

$$\frac{\partial u}{\partial y} = \lim_{y\to 0} \frac{u(0,y) - u(0,0)}{y}$$

$$= \lim_{y\to 0} \frac{0-0}{y} = 0$$

$$\frac{\partial v}{\partial x} = \lim_{x\to 0} \frac{v(x,0) - v(0,0)}{x}$$

$$= \lim_{x\to 0} \frac{0-0}{x} = 0$$

$$\frac{\partial v}{\partial y} = \lim_{y\to 0} \frac{v(0,y) - v(0,0)}{y}$$

$$= \lim_{y\to 0} \frac{0-0}{y} = 0$$

Clearly $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$.

So the C-R equations are satisfied at the origin.

Now, by definition

$$f'(0) = \lim_{z\to 0} \frac{f(z) - f(0)}{z} = \lim_{z\to 0} \frac{f(z)}{z}$$

$$= \lim_{z\to 0} \frac{\sqrt{|xy|}}{x+iy}$$

If $z \to 0$ along the line $y = mx$, we get

$$f'(0) = \lim_{x\to 0} \frac{\sqrt{|x.mx|}}{x+imx} = \lim_{x\to 0} \frac{\sqrt{|m|}}{1+im} = \frac{\sqrt{|m|}}{1+im}$$

Now this limit is not unique since it depends on m therefore $f'(0)$ does not exist.

Hence the function $f(z) = \sqrt{|xy|}$ is not regular at the origin, although C-R equations are satisfied.

1. 10. Differentiation Formulas: We have already defined the derivative of $w = f(z)$ to be

$$\frac{dw}{dz} = f'(z) = \lim_{\Delta z\to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

This definition is identical in form to that of the derivative of a function of a real variable. Hence the fundamental formulas for differentiation in the domain of complex numbers are the same as those in the case of real variables. Thus we have the following formulas:

(i) If k is a complex constant, then $\frac{d}{dz}(k) = 0$.

(ii) If k is a complex constant and w is a differentiable function, $\frac{d}{dz}(kw) = k\frac{dw}{dz}$.

(iii) If $w_1(z)$ and $w_2(z)$ are two differentiable functions, then $\frac{d}{dz}(w_1 \mp w_2) = \frac{dw_1}{dz} \mp \frac{dw_2}{dz}$.

(iv) $\frac{d}{dz}(w_1.w_2) = w_1.\frac{dw_2}{dz} + w_2.\frac{dw_1}{dz}$

(v) $\frac{d}{dz}\left(\frac{w_1}{w_2}\right) = \frac{w_2.\frac{dw_1}{dz} - w_1.\frac{dw_2}{dz}}{w_2^2}$

(vi) If w is a function of $w_1(z)$, $\frac{dw}{dz} = \frac{dw}{dw_1}.\frac{dw_1}{dz}$

(vii) If n is a positive integer, $\frac{d}{dz}(z^n) = n.z^{n-1}$. This can be extended to the case when n is a negative integer or any fraction.

EX. 18. Find where the function $w = f(z) = \frac{1}{z}$ ceases to be analytic.

Solution: Given that $w = f(z) = \frac{1}{z}$

$$\frac{dw}{dz} = \frac{d}{dz}\left(\frac{1}{z}\right) = -\frac{1}{z^2} \ if \ z \neq 0$$

For $z = 0$, $\frac{dw}{dz}$ does not exist. So, w is analytic everywhere except at the point $z = 0$ which is singular point of $f(z)$.

EX. 19. Show that an analytic function with constant real part is constant and an analytic function with constant modulus is also constant.

Solution: Let $w = f(z) = u + i\,v$ be an analytic function.

(a) Let $u(x, y) = a\ constant = c_1$

Then $\frac{\partial u}{\partial x} = 0$ and $\frac{\partial u}{\partial y} = 0$

Since w is analytic, the C-R equations are satisfied.

Therefore $\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = 0$ and $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = 0$

Since $\frac{\partial v}{\partial x} = 0$ and $\frac{\partial v}{\partial y} = 0$, it is clear that v is independent of x and y.

$$i.e., v(x, y) = a\ constant = c_2$$

Hence $w = f(z) = u + i\,v = c_1 + i\,c_2 = a\ constant$

(b) $|f(z)| = |u + i\,v| = \sqrt{u^2 + v^2} = a\ constant$

$$i.e., u^2 + v^2 = constant = c \qquad (1)$$

Differentiating (1) partially with respect to $x\ and\ y$, we get

$$2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x} = 0\ and\ 2u\frac{\partial u}{\partial y} + 2v\frac{\partial v}{\partial y} = 0$$

$$i.e., u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x} = 0\ and\ u\frac{\partial u}{\partial y} + v\frac{\partial v}{\partial y} = 0$$

Using the C-R equations , we get

$$u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x} = 0\ and\ v\frac{\partial u}{\partial x} - u\frac{\partial v}{\partial x} = 0$$

The determinant of the coefficients of $\frac{\partial u}{\partial x}$ and $\frac{\partial v}{\partial x}$ of the above two equations is

$$= \begin{vmatrix} u & v \\ v & -u \end{vmatrix} = -(u^2 + v^2) = -c$$

and this is not equal to zero

Hence the solutions of the above two equations are

$$\frac{\partial u}{\partial x} = 0 \text{ and } \frac{\partial v}{\partial x} = 0$$

Since $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}\ and\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$, we have

$$\frac{\partial u}{\partial y} = 0 \text{ and } \frac{\partial v}{\partial y} = 0$$

Therefore both u and v are independent of x and y.

$$u(x, y) = a \text{ and } v(x, y) = b$$

Hence $w = f(z) = u + i\,v = a + i\,b$ = a constant.

EX. 20. Show that an analytic function with constant argument is argument.

Solution: Let $w = f(z) = u + i\,v$ be analytic function and θ its argument.

We know that $tan\theta = \frac{v}{u}$

As θ is constant, $tan\theta$ is also constant.

Therefore $\frac{v}{u} = constatnt = k, k$ is real

Therefore $v = ku \qquad (1)$

Differentiating (1) partially with respect to x and y, we get

$$\frac{\partial v}{\partial x} = k\frac{\partial u}{\partial x} \qquad (2)$$

$$\text{and } \frac{\partial v}{\partial y} = k\frac{\partial u}{\partial y} \qquad (3)$$

But u and v are C − R equations, $i.e.,\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ (4)

$$\text{and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \qquad (5)$$

putting (3) in (4), we get

$$\frac{\partial u}{\partial x} = k\frac{\partial u}{\partial y} = -k\frac{\partial v}{\partial x} \text{ using (5)}$$

$$= -k.k\frac{\partial u}{\partial x} \text{ using (2)}$$

$$i.e.,(1+k^2)\frac{\partial u}{\partial x} = 0$$

$$1+k^2 \neq 0, so\ \frac{\partial u}{\partial x} = 0$$

$\therefore u$ is independent of x (6)

Putting $\frac{\partial u}{\partial x} = 0$ in (2), we get $\frac{\partial v}{\partial x} = 0$

Putting $\frac{\partial v}{\partial x} = 0$ in (5), we get $\frac{\partial u}{\partial y} = 0$

$\therefore u$ is independent of y (7)

From (6) and (7) we have u is a constant, independent of x and y.

Since $v = ku$

Therefore v is also a constant, independent of x and y.

Therefore $w = f(z) = u + i\,v = a\ constant.$

1.11. Properties of Analytic Functions:

Property 1. Both the real part and the imaginary part of any analytic function satisfy Laplace's equation

$$\frac{\partial^2 \emptyset}{\partial x^2} + \frac{\partial^2 \emptyset}{\partial y^2} = 0$$

Proof: Let $f(z) = u + i\,v$ be analytic in some domain of the z-plane.

Then u and v satisfy the C-R equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad (1)$$

$$\text{and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \qquad (2)$$

Differentiating (1) with respect to x and (2) with respect to y partially, we get

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)=\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right) \text{and} \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)=\frac{\partial}{\partial y}\left(-\frac{\partial v}{\partial x}\right)$$

$$\boldsymbol{i.e.}, \frac{\partial^2 u}{\partial x^2}=\frac{\partial^2 v}{\partial x \partial y} \qquad (3)$$

$$\text{and } \frac{\partial^2 u}{\partial y^2}=-\frac{\partial^2 v}{\partial y \partial x} \qquad (4)$$

Adding (3) and (4), we get

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=\frac{\partial^2 v}{\partial x \partial y}-\frac{\partial^2 v}{\partial y \partial x}$$

$$\therefore \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0 \qquad (5) \left(\text{since } \frac{\partial^2 v}{\partial x \partial y}=\frac{\partial^2 v}{\partial y \partial x}\right)$$

Similarly we can show that

$$\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=0 \qquad (6)$$

(5) and (6) shows that u and v satsfy the Laplace's equation

$$\frac{\partial^2 \emptyset}{\partial x^2}+\frac{\partial^2 \emptyset}{\partial y^2}=0 \qquad (7)$$

which is Laplace's partial differential equation in the two independent variables x and y. This equation occurs frequently in mathematical physics. It is satisfied by the potential at a point not occupied by matter in a two-dimensional gravitational field. It is also satisfied by the velocity potential and stream function of two-dimensional irrotational flow of an incompressible non-viscous fluid.

Note: In proving results (5) and (6), it has assumed that the second order partial derivatives of u and v with respect to x and y all exist and further are continuous.

Any function which possesses continuous second order partial derivatives and which satisfies Laplace's equation is called a ***harmonic function***. Two harmonic functions, u and v which are such that $u + i\, v$ is an analytic function are called ***conjugate harmonic functions***.

The importance of analytic function of a complex variable is that such a function furnishes us with distinct solutions of Laplace's equation. It is this connection of analytic function with Laplace's equation that has given a great importance to the theory of functions of a complex variable in applied mathematics.

The equation (7) is written as $\nabla^2 \emptyset = 0$, where

$$\nabla^2 \equiv \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}$$

∇^2 is called the Laplacian operator.

Property 2. If $w = f(z) = u + i\,v$ is an analytic function, the curves of the family $u(x, y) = constant = c_1$ cut orthogonally the curves of the family $v(x, y) = \ constant = c_2$.

Proof: Given that $w = f(z) = u + i\,v$ is an analytic function

Then u and v are satisfy C-R equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad (1)$$

$$\text{and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \qquad (2)$$

Suppose $u(x, y) = c_1$ is the equation of the family of curves for different values of c_1. Similarly, $v(x, y) = c_2$ is the equation of the family of curves for different values of c_2.

Since $u(x, y) = c_1$, by the total differentiation,

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy$$

$$\text{Hence } \frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dx} = 0 \text{ since } u = c_1$$

$$\text{So } \frac{dy}{dx} = -\frac{\left(\frac{\partial u}{\partial x}\right)}{\left(\frac{\partial u}{\partial y}\right)} \qquad (3)$$

This is the slope of the general curve of the u-family.

Similarly for the v –family,

$$\frac{dy}{dx} = -\frac{\left(\frac{\partial v}{\partial x}\right)}{\left(\frac{\partial v}{\partial y}\right)} \qquad (4)$$

Using (1) and (2), (4) can be written as

$$\frac{dy}{dx} = \frac{\left(\frac{\partial u}{\partial y}\right)}{\left(\frac{\partial u}{\partial x}\right)}$$

The product of the slopes of the two families is

$$= -\frac{\left(\frac{\partial u}{\partial x}\right)}{\left(\frac{\partial u}{\partial y}\right)} X \frac{\left(\frac{\partial u}{\partial y}\right)}{\left(\frac{\partial u}{\partial x}\right)} = -1.$$

Hence the curves cut each other orthogonally. The two families are said to be the ***orthogonal trajectories*** of one another.

Result 1. To prove that $\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}=4\frac{\partial^2}{\partial z\partial \bar{z}}$

Proof: Let $z=x+iy$ *and* $\bar{z}=x-iy$ so that

$$x=\frac{z+\bar{z}}{2}\ and\ y=\frac{z-\bar{z}}{2i}=-\frac{i(z-\bar{z})}{2}$$

This implies $\frac{\partial x}{\partial z}=\frac{1}{2}=\frac{\partial x}{\partial \bar{z}},\frac{\partial y}{\partial z}=-\frac{i}{2}=-\frac{\partial y}{\partial \bar{z}}$

Let $f=f(x,y)$. Then $f=f(z,\bar{z})$

We have

$$\frac{\partial f}{\partial z}=\frac{\partial f}{\partial x}\frac{\partial x}{\partial z}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial z}$$

$$=\frac{1}{2}\left(\frac{\partial f}{\partial x}-\ i\frac{\partial f}{\partial y}\right)$$

$$\frac{\partial f}{\partial \bar{z}}=\frac{\partial f}{\partial x}\frac{\partial x}{\partial \bar{z}}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial \bar{z}}$$

$$=\frac{1}{2}\left(\frac{\partial f}{\partial x}+\ i\frac{\partial f}{\partial y}\right)$$

Now $\frac{\partial^2 f}{\partial z\partial \bar{z}}=\frac{\partial}{\partial z}\left(\frac{\partial f}{\partial \bar{z}}\right)$

$$=\frac{1}{4}\left(\frac{\partial}{\partial x}-\ i\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x}+\ i\frac{\partial}{\partial y}\right)f$$

$$=\frac{1}{4}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)f$$

$$\therefore \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}=4\frac{\partial^2}{\partial z\partial \bar{z}}$$

Result 2. If $f(z)$ is a regular function of z; Prove that

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^2=4|f'(z)|^2$$

Proof: Recall that $\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}=4\frac{\partial^2}{\partial z\partial \bar{z}}$

$$\Rightarrow \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^2=4\frac{\partial^2}{\partial z\partial \bar{z}}\ f(z)f(\bar{z})\ as\ |z|^2=z\,\bar{z}$$

$$=4\frac{\partial}{\partial z}\left[\frac{\partial}{\partial \bar{z}}\ f(z)f(\bar{z})\right]$$

$$=4\frac{\partial}{\partial z}[\ f(z)f'(\bar{z})]$$

$$= 4f'(z)f'(\bar{z}) = 4|f'(z)|^2$$

(since f(z) is treated as constant in differentiating with respect to $\bar{z}$)

Result 3. If $w = f(z)$ is a regular function of z such that $f'(z) \neq 0$. Prove that

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) log\ |f'(z)| = 0$$

If $|f'(z)|$ is the product of a function of x and function of y, then show that $f'(z) = e^{\alpha z^2 + \beta z + \gamma}$ where α is the real and β, γ are complex constants.

Proof: Recall that $\quad \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4\frac{\partial^2}{\partial z \partial \bar{z}}$

$$\Rightarrow \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) log\ |f'(z)| = 4\frac{\partial^2}{\partial z \partial \bar{z}}\ log\ |f'(z)|$$

$$= 2\frac{\partial^2}{\partial z \partial \bar{z}}\ log\ |f'(z)|^2$$

$$= 2\frac{\partial^2}{\partial z \partial \bar{z}}\ log\{f'(z)f'(\bar{z})\}\ \ as\ |z|^2 = z\,\bar{z}$$

$$= 2\frac{\partial^2}{\partial z \partial \bar{z}}\ log\{f'(z)\} + 2\frac{\partial^2}{\partial z \partial \bar{z}}\ log\{f'(\bar{z})\}$$

$$= 2\frac{\partial}{\partial \bar{z}}\left\{\frac{f''(z)}{f'(z)}\right\} + 2\frac{\partial}{\partial z}\left\{\frac{f''(\bar{z})}{f'(\bar{z})}\right\}$$

$$= 0 + 0 = 0$$

It follows from the fact that $f(z)$ is treated as constant in differentiation with respect to $\bar{z}$.

$$\text{Hence } \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) log\ |f'(z)| = 0 \qquad (1)$$

Let $|f'(z)| = \emptyset(x).\psi(y)$

From (1),

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) log\ (\emptyset(x).\psi(y)) = 0$$

$$\frac{\partial^2}{\partial x^2}[log\ \emptyset(x) + log\ \psi(y)\ ] + \frac{\partial^2}{\partial y^2}[log\ \emptyset(x) + log\ \psi(y)\ ] = 0$$

$$\frac{\partial^2}{\partial x^2}[log\ \emptyset(x)] + \frac{\partial^2}{\partial y^2}[log\ \psi(y)\ ] = 0$$

$$\frac{d^2}{dx^2}[log\ \emptyset(x)] + \frac{d^2}{dy^2}[log\ \psi(y)\ ] = 0$$

$$\frac{d^2}{dx^2}[log\ \emptyset(x)] = -\frac{d^2}{dy^2}[log\ \psi(y)\ ] = 2p, say$$

For L.H.S. and R.H.S both are independent of each other.

$\frac{d^2}{dx^2}[log\ \emptyset(x)] = 2p,$ given an integration

$$\frac{d}{dx}[log\ \emptyset(x)] = 2px + q$$

Again integrating, $log\ \emptyset(x) = px^2 + qx + r$

Similarly, $-log\ \psi(y) = py^2 + q_1 y + r_1$

$$log\ (\emptyset(x).\psi(y)) = log\ \emptyset(x) + log\ \psi(y)$$

$$= px^2 + qx + r - py^2 - q_1 y - r_1$$

$$= p(\ x^2 - y^2) + (qx - q_1 y) + (r - r_1)$$

or $\quad |f'(z)| = \emptyset(x).\psi(y)$

$$= exp\ [p(\ x^2 - y^2) + (qx - q_1 y) + (r - r_1)] \qquad (2)$$

Now $|exp(\alpha z^2 + \beta z + \gamma)| = |exp\{\alpha(x + iy)^2 + \beta(x + iy) + \gamma\}|$

$$= |exp\{[\alpha(x^2 - y^2) + 2i\alpha xy] + (a + ib)(x + iy) + (c + id)\}|$$

as α is a real.

$$= |exp\{\alpha(x^2 - y^2) + ax - by + c\} + exp\{i(2\alpha xy + bx + ay + d)\}|$$

$$= |exp\{\alpha(x^2 - y^2) + ax - by + c\}|$$

As $\left|e^{ip}\right| = 1$ for any real p, which of the same form as (2).

Hence we can write

$$f'(z) = \ exp(\alpha z^2 + \beta z + \gamma)$$

Result 4. If $f(z)$ is an analytic function of z, prove that

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|Rf(z)|^2 = 2|f'(z)|^2$$

Proof: Let $f(z) = u + iv$, then $Rf(z) = u$.

$$\frac{\partial}{\partial x}(u^2) = 2u\frac{\partial u}{\partial x}$$

$$\frac{\partial^2}{\partial x^2}(u^2) = 2\left[\left(\frac{\partial u}{\partial x}\right)^2 + u\frac{\partial^2 u}{\partial x^2}\right] \qquad (1)$$

Similarly,

$$\frac{\partial^2}{\partial y^2}(u^2) = 2\left[\left(\frac{\partial u}{\partial y}\right)^2 + u\frac{\partial^2 u}{\partial y^2}\right] \qquad (2)$$

Adding (1) and (2), we get

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|Rf(z)|^2 = 2\left[\left(\frac{\partial u}{\partial x}\right)^2+u\frac{\partial^2 u}{\partial x^2}\right]+2\left[\left(\frac{\partial u}{\partial y}\right)^2+u\frac{\partial^2 u}{\partial y^2}\right]$$

$$= 2\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2+u\left\{\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right\}\right] \qquad (3)$$

But u satisfies Laplace's equation and $f(z)$ is an analytic function, u and v satisfies C-R equations, that is

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}\ and\ \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$$

(3) becomes

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|Rf(z)|^2 = 2\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2\right] \qquad (4)$$

But $f'(z)=\frac{\partial w}{\partial x}=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}$ then $f'(\bar{z})=\frac{\partial u}{\partial x}-i\frac{\partial v}{\partial x}$

$$|f'(z)|^2 = f'(z)f'(\bar{z}) = \left(\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}\right)\left(\frac{\partial u}{\partial x}-i\frac{\partial v}{\partial x}\right)=\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2$$

In view of this, the last gives

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|Rf(z)|^2 = 2|f'(z)|^2$$

Result 5. If $u(x,y)$ and $v(x,y)$ are harmonic functions in a region R, prove that

$$\left(\frac{\partial u}{\partial y}-\frac{\partial v}{\partial x}\right)+i\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)$$

is an analytic function of $z=x+iy$.

Proof: Suppose $u(x,y)$ and $v(x,y)$ satisfy Laplace's equation.

$$i.e.,\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0 \qquad (1)$$

$$and\ \frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=0 \qquad (2)$$

Also suppose

$$s=\frac{\partial u}{\partial y}-\frac{\partial v}{\partial x}\ and\ t=\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}$$

To prove that $s+it$ is an analytic function, we have to show that

(i) $\frac{\partial s}{\partial x}=\frac{\partial t}{\partial y}\ and\ \frac{\partial s}{\partial y}=-\frac{\partial t}{\partial x}$

(ii) $\frac{\partial s}{\partial x},\frac{\partial t}{\partial y},\frac{\partial s}{\partial y}\ and\ -\frac{\partial t}{\partial x}$ are continuous

$$\frac{\partial s}{\partial x}-\frac{\partial t}{\partial y}=-\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)=0 \qquad (3)\ from\ (2)$$

$$\frac{\partial s}{\partial y}+\frac{\partial t}{\partial x}=\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)=0 \qquad (4)\ from\ (1)$$

From (3) and (4), the result (i) follows.

Existence of (1) and (2) implies that the result (ii).

1.12. Construction of an Analytic Function whose Real or Imaginary Part is known:

Let $f(z)=u+iv$ be an analytic function, whose real part u alone is known beforehand. We can find v, the imaginary part and also the function $f(z)$. The procedure is as follows:

First Method:

$$dv=\frac{\partial v}{\partial x}dx+\frac{\partial v}{\partial y}dy$$

But $u\ and\ v$ are satisfy C-R equations

$$i.e., \frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}\ and\ \frac{\partial v}{\partial x}=-\frac{\partial u}{\partial y}$$

Hence

$$dv=-\frac{\partial u}{\partial y}dx+\frac{\partial u}{\partial x}dy \qquad (1)$$

Now $\frac{\partial}{\partial y}\left(-\frac{\partial u}{\partial y}\right)=-\frac{\partial^2 u}{\partial y^2}$ and $\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)=\frac{\partial^2 u}{\partial x^2}$

As u satisfies Laplace's equation,

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$$

$$i.e., \frac{\partial^2 u}{\partial x^2}=-\frac{\partial^2 u}{\partial y^2}$$

Hence

$$\frac{\partial}{\partial y}\left(-\frac{\partial u}{\partial y}\right)=\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)$$

and so the R.H.S. of (1) is a perfect differential.

Also $\frac{\partial u}{\partial y}$ and $\frac{\partial u}{\partial x}$ are known, since u is given.

Hence integrating (1), $v=\int\left(-\frac{\partial u}{\partial y}dx+\frac{\partial u}{\partial x}dy\right)+c$

where c is an arbitrary constant. Thus v is known and the function $f(z)=u+iv$ is determined.

Second Method:

We know that $f'(z) = \frac{\partial w}{\partial x} = \frac{\partial}{\partial x}(u+iv) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$ (1)

$\left(\text{since } \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \text{ from C} - \text{R equations}\right)$

Since u is given, $\frac{\partial u}{\partial y}$ and $\frac{\partial u}{\partial x}$ are known.

Hence integrating (1), $f(z) = \int \left(\frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}\right) dz + c.$

It is implied that the expression

$$\frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$$

must be expressed in terms of $z = x + iy$, and then the above integration is to be effected.

Third Method (Milne-Thomson Method):

To find $f(z)$, when the real part $u(x,y)$ is given.

Let $f(z) = u(x,y) + i\,v(x,y)$ (1)

Since $z = x + iy, \bar{z} = x - iy$, we have

$$x = \frac{z+\bar{z}}{2}, y = \frac{z-\bar{z}}{2i}$$

So $f(z) = u\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right) + i\,v\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right)$

Consider this as a formal identity in the two independent variables $z, \bar{z}$.

Putting $\bar{z} = z$, we get

$$f(z) = u(z,0) + i\,v(z,0) \qquad (2)$$

(1) Is the same as (1), if we replace $x \; by \; z \; and \; y \; by \; 0$.

Now $f'(z) = \frac{\partial w}{\partial x} = \frac{\partial}{\partial x}(u+iv) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$

$\left(\text{since } \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \text{ from C} - \text{R equations}\right)$

Let $\frac{\partial u}{\partial x} = \emptyset_1(x,y)$and $\frac{\partial u}{\partial y} = \emptyset_2(x,y)$.

Then $f'(z) = \emptyset_1(x,y) - i\,\emptyset_2(x,y)$ (3)

Now, to express $f'(z)$ completely in terms of z, we replace $x \; by \; z$ and y by 0 in the expression (3).

$$f'(z) = \emptyset_1(z,0) - i\,\emptyset_2(z,0)$$

Hence $f(z)=\int\left(\emptyset_1(z,0)-i\,\emptyset_2(z,0)\right)dz\pm C$

Similarly, given the imaginary part v, we can find u such that $u+iv$ is analytic. Let us use Milne-Thomson Method.

Now $f'(z)=\frac{\partial w}{\partial x}=\frac{\partial}{\partial x}(u+iv)=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}=\frac{\partial v}{\partial y}+i\frac{\partial v}{\partial x}$

$$\left(\text{since } \frac{\partial u}{\partial x}=\frac{\partial v}{\partial y} \text{ from C}-\text{R equations}\right)$$

$f'(z)=\psi_1(x,y)-i\,\psi_2(x,y)$

where $\frac{\partial v}{\partial y}=\psi_1(x,y)$ and $\frac{\partial v}{\partial x}=\psi_2(x,y)$

Now, to express $f'(z)$ completely in terms of z, we replace $x\ by\ z$ and y by 0 in the above expression

$$f'(z)=\psi_1(z,0)-i\,\psi_2(z,0)$$

Hence $$f(z)=\int\left(\psi_1(z,0)-i\,\psi_2(z,0)\right)dz\pm C.$$

1.13. The Complex Potential Function:

We have seen that every analytic function $f(z)=u(x,y)+i\,v(x,y)$ defines two families of curves

$$u(x,y)=c_1 \text{ and } v(x,y)=c_2$$

which form an orthogonal system. This property of analytic functions is of great use in field and flow problems. We consider two dimensional regions in which there is sort of steady flow like fluid flow, heat flow or electric current flow. The paths of fluid particles are called stream lines and their orthogonal trajectories are termed as equi-potentials.

In physical applications, the analytic function

$$w=\emptyset(x,y)+i\,\psi(x,y)$$

is referred to as the complex potential function. Its real part $\emptyset(x,y)$ represents the velocity potential function and the imaginary part $\psi(x,y)$ represents the stream function. Both $\emptyset$ and ψ will satisfy Laplace's equation and given any one of them, we can find the other.

Also the magnitude of the fluid velocity v or the electric intensity

$$E\ is\ \left|\frac{dw}{dz}\right|$$

EX. 21. Show that the function $u=\frac{1}{2}log(x^2+y^2)$ is harmonic and determine its conjugate.

Solution: Given $u=\frac{1}{2}log(x^2+y^2)$

We have $\frac{\partial u}{\partial x} = \frac{x}{x^2+y^2}, \frac{\partial u}{\partial y} = \frac{y}{x^2+y^2}$

Now $\frac{\partial^2 u}{\partial x^2} = \frac{(x^2+y^2).1 - x.2x}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$

Similarly, $\frac{\partial^2 u}{\partial y^2} = \frac{x^2-y^2}{(x^2+y^2)^2}$

Clearly, $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$

Hence u is harmonic. Let v be the conjugate of u. Then

$$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy$$

$$= -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy \text{ (using } C-R \text{ equations)}$$

$$= \frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy$$

$$= \frac{xdy - ydx}{x^2+y^2}$$

$$= \frac{x^2}{x^2+y^2}\left(\frac{xdy - ydx}{x^2}\right)$$

$$= \frac{x^2}{x^2+y^2}d\left(\frac{y}{x}\right)$$

$$= \frac{1}{1+\left(\frac{y}{x}\right)^2}d\left(\frac{y}{x}\right) \quad = d\left(tan^{-1}\frac{y}{x}\right)$$

Hence Integrating

$$v = tan^{-1}\frac{y}{x} + C$$

EX. 22. Find the analytic function whose real part is $\frac{x}{x^2+y^2}$.

Solution: Let $f(z) = u + iv$ where $u = \frac{x}{x^2+y^2}$

We have $\frac{\partial u}{\partial x} = \frac{y^2-x^2}{(x^2+y^2)^2}, \frac{\partial u}{\partial y} = -\frac{2xy}{(x^2+y^2)^2}$

Now

$$f'(z) = \frac{\partial w}{\partial x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$$

$$= \frac{y^2-x^2}{(x^2+y^2)^2} + i\frac{2xy}{(x^2+y^2)^2}$$

$$= \frac{(y+ix)^2}{(x^2+y^2)^2} = \frac{(y+ix)^2}{(y+ix)^2(y-ix)^2}$$

$$= \frac{1}{(y-ix)^2} = \frac{i^2}{(iy+x)^2} = -\frac{1}{z^2}$$

Integrating, $f(z) = \frac{1}{z} + c$

EX. 23. Find the analytic function whose imaginary part is $e^x(x\,sin\,y + y\,cos\,y)$.

Solution: Let $f(z) = u + iv$ where $v = e^x(x\,sin\,y + y\,cos\,y)$

We have $\frac{\partial v}{\partial x} = e^x(\,sin\,y + x\,sin\,y + y\,cos\,y)$

and $\frac{\partial v}{\partial y} = e^x(x\,cos\,y + cos\,y - y\,sin\,y)$

Now $f'(z) = \frac{\partial w}{\partial x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x}$

$$= e^x(x\,cos\,y + cos\,y - y\,sin\,y) + ie^x(\,sin\,y + x\,sin\,y + y\,cos\,y)$$

By Milne-Thomson's method, $f'(z)$ is expressed in terms of z by replacing x by z and y by 0.

Hence $f'(z) = e^z(z+1)$.

Integrating, $f(z) = (z+1)e^z - e^z + c = ze^z + c$

where c is a complex constant.

EX. 24. Find the analytic function $f(z) = u + iv$ of which the real part

$$u = e^x(x\,cos\,y - y\,sin\,y)$$

Solution: Given

$$u = e^x(x\,cos\,y - y\,sin\,y)$$

We have

$$\frac{\partial u}{\partial x} = e^x(\,cos\,y + x\,cos\,y - y\,sin\,y)$$

$$\text{and } \frac{\partial u}{\partial y} = e^x(-x\,sin\,y - sin\,y - y\,cos\,y)$$

Now

$$f'(z) = \frac{\partial w}{\partial x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$$

$$= e^x(\,cos\,y + x\,cos\,y - y\,sin\,y) - ie^x(-x\,sin\,y - sin\,y - y\,cos\,y)$$

By Milne-Thomson's method, $f'(z)$ is expressed in terms of z by replacing x by z and y by 0.

Hence

$$f'(z) = e^z(z+1)$$

Integrating, $f(z) = (z+1)e^z - e^z + c = ze^z + c$

where c is a complex constant.

EX. 25. Find the analytic function $f(z) = u + iv$ of which the real part

$$u = \frac{\sin 2x}{\cosh 2y - \cos 2x}$$

Solution: Let $f(z) = u + iv$, where

$$u = \frac{\sin 2x}{\cosh 2y - \cos 2x}$$

We have

$$\frac{\partial u}{\partial x} = \frac{(\cosh 2y - \cos 2x).2\cos 2x - \sin 2x.(2\sin 2x)}{(\cosh 2y - \cos 2x)^2}$$

$$\text{and } \frac{\partial u}{\partial y} = \frac{\sin 2x.(2\sinh 2y)}{(\cosh 2y - \cos 2x)^2}$$

Now $f'(z) = \dfrac{\partial w}{\partial x} = \dfrac{\partial u}{\partial x} + i\dfrac{\partial v}{\partial x} = \dfrac{\partial u}{\partial x} - i\dfrac{\partial u}{\partial y}$

$$= \frac{(\cosh 2y - \cos 2x).2\cos 2x - \sin 2x.(2\sin 2x)}{(\cosh 2y - \cos 2x)^2} - i\frac{\sin 2x.(2\sinh 2y)}{(\cosh 2y - \cos 2x)^2}$$

By Milne-Thomson's method, $f'(z)$ is expressed in terms of z by replacing x by z and y by 0.

Hence

$$f'(z) = \frac{2\cos 2z - 2}{(1 - \cos 2z)^2} = \frac{-2}{1 - \cos 2z} = \frac{-2}{2\sin^2 z} = -\operatorname{cosec}^2 z$$

Integrating,

$$f(z) = \cot z + ic$$

Taking the constant of integration as imaginary. Since u does not contain any constant.

EX. 26. An incompressible fluid flowing over the xy-plane has the velocity potential

$$\emptyset = x^2 - y^2 + \frac{x}{x^2 + y^2}$$

Examine if this is possible and find a stream function $\boldsymbol{\psi}$.

Solution: Given

$$\emptyset = x^2 - y^2 + \frac{x}{x^2 + y^2} \qquad (1)$$

Then

$$\frac{\partial\emptyset}{\partial x} = 2x + \frac{(x^2+y^2).1 - x.2x}{(x^2+y^2)^2} = 2x + \frac{y^2 - x^2}{(x^2+y^2)^2} \qquad (2)$$

$$\frac{\partial^2\emptyset}{\partial x^2} = 2 + \frac{(x^2+y^2)^2.(-2x) - (y^2-x^2).2(x^2+y^2).2x}{(x^2+y^2)^4}$$

$$= 2 - \frac{2x(x^2+y^2+2y^2-2x^2)}{(x^2+y^2)^3}$$

$$= 2 - \frac{2x(3y^2-x^2)}{(x^2+y^2)^3} \qquad (3)$$

$$\frac{\partial\emptyset}{\partial y} = -2y - \frac{x.2y}{(x^2+y^2)^2} = -2y - \frac{2xy}{(x^2+y^2)^2} \qquad (4)$$

$$\frac{\partial^2\emptyset}{\partial y^2} = -2 - 2x\left[\frac{(x^2+y^2)^2.1 - y.2(x^2+y^2).2y}{(x^2+y^2)^4}\right]$$

$$= -2 - \frac{2x.(x^2-3y^2)}{(x^2+y^2)^3} \, (5)$$

Clearly $\dfrac{\partial^2\emptyset}{\partial x^2} + \dfrac{\partial^2\emptyset}{\partial y^2} = 0.$

That is $\emptyset$ satisfies Laplace's equation.

Hence it can be a possible form of the velocity potential function.

To find the stream function $\boldsymbol{\psi}$, we know that $\emptyset + i\boldsymbol{\psi}$ is analytic.

Therefore $\emptyset$ and $\boldsymbol{\psi}$ satisfy C-R equations.

$$ie., \frac{\partial\psi}{\partial y} = \frac{\partial\emptyset}{\partial x} \qquad (6)$$

$$and \ \frac{\partial\psi}{\partial x} = -\frac{\partial\emptyset}{\partial y} \qquad (7)$$

Taking (7) and using the result given by (4), we have

$$\frac{\partial\psi}{\partial x} = 2y + \frac{2xy}{(x^2+y^2)^2} \qquad (8)$$

Integrating with respect to x, we get

$$\psi = 2xy + \int \frac{2xy}{(x^2+y^2)^2} dx + F(y)$$

where $F(y)$ is an arbitrary function of y.

$$i.e., \psi = 2xy - \frac{y}{x^2 + y^2} + F(y) \qquad (9)$$

Differentiating (9) with respect to y, we get

$$\frac{\partial\psi}{\partial y} = 2x - \left[\frac{(x^2 + y^2).1 - y.2y}{(x^2 + y^2)^2}\right] + F'(y)$$

$$= 2x + \frac{y^2 - x^2}{(x^2 + y^2)^2} + F'(y) \qquad (10)$$

But from (6) and (2), we have

$$\frac{\partial\psi}{\partial y} = \frac{\partial\emptyset}{\partial x} = 2x + \frac{y^2 - x^2}{(x^2 + y^2)^2} \qquad (11)$$

Comparing (10) and (11), we get

$$F'(y) = 0$$

$i.e., F(y) = C,$ an arbitrary constant.

Hence from (9), $\psi = 2xy - \frac{y}{x^2+y^2} + C.$

Taking $C = 0$, we get the simplest form of the stream function

$$\psi = 2xy - \frac{y}{x^2 + y^2}$$

EX. 27. In a two dimensional fluid flow the stream function is

$$\psi = -\frac{y}{x^2 + y^2}$$

Find the velocity potential $\emptyset$.

Solution: Given

$$\psi = -\frac{y}{x^2 + y^2}$$

is a stream function, i.e., it must satisfy Laplace's equation, $\nabla^2\psi = 0$

Now

$$\psi = -\frac{y}{x^2 + y^2}$$

We have

$$\frac{\partial\psi}{\partial x} = \frac{2xy}{(x^2 + y^2)^2}$$

$$\frac{\partial\psi}{\partial y} = \frac{(y^2 - x^2)}{(x^2 + y^2)^2}$$

$$\frac{\partial^2\psi}{\partial x^2} = \frac{2y.(y^2 - 3x^2)}{(x^2 + y^2)^3}$$

$$\frac{\partial^2\psi}{\partial y^2} = -\frac{2y.(y^2 - 3x^2)}{(x^2 + y^2)^3}$$

$$\therefore \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = 0 \; or \; \nabla^2\psi = 0$$

Hence ψ satisfy Laplace's equation.

Now $\emptyset$ is the velocity potential, let $w = \emptyset + i\psi$

$$\frac{dw}{dz} = \frac{\partial w}{\partial x} = \frac{\partial \emptyset}{\partial x} + i\frac{\partial \psi}{\partial x} = \frac{\partial \psi}{\partial y} + i\frac{\partial \psi}{\partial x}$$

$$= \frac{(y^2 - x^2)}{(x^2 + y^2)^2} + i\frac{2xy}{(x^2 + y^2)^2}$$

By Milne-Thomson's method, $f'(z)$ is expressed in terms of z by replacing x by z and y by 0.

Hence

$$\frac{dw}{dz} = f'(z) = \frac{-1}{z^2}$$

Integrating,

$$w = f(z) = \frac{1}{z} + c$$

Where c is a complex constant.

$$\emptyset + i\psi = \frac{1}{z} + c = \frac{1}{x + iy} + c$$

$$= \frac{x - iy}{x^2 + y^2} + c$$

$$= \frac{x - iy}{x^2 + y^2} + a + ib, where\ c = a + ib$$

Equating real parts on both sides, we get

Velocity potential $= \emptyset = \frac{x}{x^2+y^2} + a.$

EX. 28. If $f(z) = u + iv$ is analytic function and $u - v = e^x(\cos y - \sin y)$, find $f(z)$ in terms of z.

Solution: Let $f(z) = u + iv$ (1)

So that $if(z) = iu - v$ (2)

Adding (1) and (2), we get

$$(1 + i)f(z) = (u - v) + i(u + v)$$

$$i.e., F(z) = U + iV \qquad (3)$$

Where $U = u - v, V = u + v$ and $F(z) = (1 + i)f(z)$

If $f(z)$ is analytic, then $F(z)$ is also analytic.

Given $U = u - v = e^x(\cos y - \sin y)$

We have

$$\frac{\partial U}{\partial x} = e^x(\cos y - \sin y)$$

$$\frac{\partial U}{\partial y} = e^x(-\cos y - \sin y)$$

Now $F'(z) = \frac{\partial U}{\partial x} + i\frac{\partial V}{\partial x} = \frac{\partial U}{\partial x} - i\frac{\partial U}{\partial y}$

$$= e^x(\cos y - \sin y) - ie^x(-\cos y - \sin y)$$

By Milne-Thomson's method, $F'(z)$ is expressed in terms of z by replacing x by z and y by 0.

Hence

$$F'(z) == e^z(1+i)$$

Integrating,

$$F(z) = (1+i)e^z + C$$

$$i.e., (1+i)f(z) = (1+i)e^z + C$$

$f(z) = e^z + c$ where c is a complex constant.

EX. 29. If $u + v = \frac{x}{x^2+y^2}$, when $f(1) = 1$ and $f(z)$ is analytic function of z, find $f(z)$in terms of z.

Solution: Let $f(z) = u + iv$ (1)

So that $if(z) = iu - v$ (2)

Adding (1) and (2), we get

$$(1+i)f(z) = (u-v) + i(u+v)$$

$$i.e., F(z) = U + iV \quad (3)$$

Where $U = u - v, V = u + v \; and \; F(z) = (1+i)f(z)$

If $f(z)$ is analytic, then $F(z)$ is also analytic.

Here $V = u + v = \frac{x}{x^2+y^2}$

We have

$$\frac{\partial V}{\partial x} = \frac{y^2 - x^2}{(x^2+y^2)^2}$$

$$\frac{\partial V}{\partial y} = -\frac{2xy}{(x^2+y^2)^2}$$

Now $F'(z) = \frac{\partial U}{\partial x} + i\frac{\partial V}{\partial x} = \frac{\partial V}{\partial y} + i\frac{\partial V}{\partial x}$

$$= -\frac{2xy}{(x^2+y^2)^2} + i\frac{y^2 - x^2}{(x^2+y^2)^2}$$

By Milne-Thomson's method, $F'(z)$ is expressed in terms of z by replacing x by z and y by 0.

Hence

$$F'(z) = \frac{-i}{z^2}$$

Integrating,

$$F(z) = \frac{i}{z} + C$$

$$i.e., (1+i)f(z) = \frac{i}{z} + C$$

$$f(z) = \frac{i}{1+i}\frac{1}{z} + c \text{ where } c \text{ is a complex constant.}$$

EX.30. If $u + v = (x-y)(x^2+4xy+y^2)$ and $f(z)$ is analytic function of z, find $f(z)$ in terms of z.

Solution: Let $f(z) = u + iv$ (1)

So that $if(z) = iu - v$ (2)

Adding (1) and (2), we get

$$(1+i)f(z) = (u-v) + i(u+v)$$

$$i.e., F(z) = U + iV \qquad (3)$$

Where $U = u - v, V = u + v \text{ and } F(z) = (1+i)f(z)$

If $f(z)$ is analytic, then $F(z)$ is also analytic.

Here $V = u + v = (x-y)(x^2+4xy+y^2)$

We have

$$\frac{\partial V}{\partial x} = (x-y)(2x+4y) + (x^2+4xy+y^2)$$

$$\frac{\partial V}{\partial y} = (x-y)(4x+2y) - (x^2+4xy+y^2)$$

Now $F'(z) = \frac{\partial U}{\partial x} + i\frac{\partial V}{\partial x} = \frac{\partial V}{\partial y} + i\frac{\partial V}{\partial x}$

$= (x-y)(4x+2y) - (x^2+4xy+y^2) + i[(x-y)(2x+4y) + (x^2+4xy+y^2)]$

By Milne-Thomson's method, $F'(z)$ is expressed in terms of z by replacing x by z and y by 0.

Hence

$$F'(z) = 3(1+i)z^2$$

Integrating,

$$F(z) = (1+i)z^3 + C$$

$$i.e., (1+i)f(z) = (1+i)z^3 + C$$
$$f(z) = z^3 + c \text{ where } c \text{ is a complex constant.}$$

EX. 31. If $u + v = \dfrac{2 \sin 2x}{e^{2y} + e^{-2y} - 2\cos 2x}$ and $f(z)$ is analytic function of z, find $f(z)$ in terms of z.

Solution: Let $f(z) = u + iv$ (1)

So that $if(z) = iu - v$ (2)

Adding (1) and (2), we get

$$(1+i)f(z) = (u-v) + i(u+v)$$
$$i.e., F(z) = U + iV \qquad (3)$$

Where $U = u - v$, $V = u + v$ and $F(z) = (1+i)f(z)$

If $f(z)$ is analytic, then $F(z)$ is also analytic.

$$Here\ V = u + v = \frac{2 \sin 2x}{e^{2y} + e^{-2y} - 2\cos 2x} = \frac{\sin 2x}{\cosh 2y - \cos 2x}$$

We have

$$\frac{\partial V}{\partial x} = \frac{(\cosh 2y - \cos 2x).2\cos 2x - \sin 2x.2\sin 2x}{(\cosh 2y - \cos 2x)^2}$$
$$= \frac{2\cos 2x.\cosh 2y - 2}{(\cosh 2y - \cos 2x)^2}$$
$$\frac{\partial V}{\partial y} = \frac{2\sin 2x.\sinh 2y}{(\cosh 2y - \cos 2x)^2}$$
$$Now\ F'(z) = \frac{\partial U}{\partial x} + i\frac{\partial V}{\partial x} = \frac{\partial V}{\partial y} + i\frac{\partial V}{\partial x}$$
$$= \frac{2\sin 2x.\sinh 2y}{(\cosh 2y - \cos 2x)^2} + i\frac{2\cos 2x.\cosh 2y - 2}{(\cosh 2y - \cos 2x)^2}$$

By Milne-Thomson's method, $F'(z)$ is expressed in terms of z by replacing x by z and y by 0.

Hence

$$F'(z) = 2i\frac{\cos 2z - 1}{(1 - \cos 2z)^2} = \frac{-2i}{1 - \cos 2z}$$
$$= \frac{-2i}{2\sin^2 z} = -i\,cosec^2 z$$

Integrating,

$$F(z) = i\cot z + C$$
$$i.e., (1+i)f(z) = i\cot z + C$$

$$f(z) = \frac{i}{1+i}\ cot\ z + c \text{ where } c \text{ is a complex constant.}$$

EX. 32. If $u - v = \frac{\cos\ x + sin\ x - e^{-y}}{2\ cos\ x - e^{y} - e^{-y}}$, when $f\left(\frac{\pi}{2}\right) = 0$ and $f(z)$ is analytic function of z, find $f(z)$ in terms of z.

Solution: Let $f(z) = u + iv$ (1)

So that $if(z) = iu - v$ (2)

Adding (1) and (2), we get

$$(1+i)f(z) = (u - v) + i(u + v)$$

$$i.e., F(z) = U + iV \qquad (3)$$

Where $U = u - v$, $V = uv$ and $F(z) = (1+i)f(z)$

If $f(z)$ is analytic, then $F(z)$ is also analytic.

$$\text{Given } U = u - v = \frac{\cos\ x + sin\ x - e^{-y}}{2\ cos\ x - e^{y} - e^{-y}} = \frac{\cos\ x + sin\ x - e^{-y}}{2\ (cos\ x - \ cosh\ y)}$$

We have

$$\frac{\partial U}{\partial x} = \frac{(\ cos\ x - cosh\ y).(\cos\ x - sin\ x) + (\cos\ x + sin\ x - e^{-y}).(sin\ x)}{2(cos\ x - cosh\ y)^2}$$

$$\frac{\partial U}{\partial y} = \frac{(cos\ x - cosh\ y).e^{-y} + (\cos\ x + sin\ x - e^{-y}).(sinh\ y)}{2(cos\ x - cosh\ y)^2}$$

$$\text{Now } F'(z) = \frac{\partial U}{\partial x} + i\frac{\partial V}{\partial x} = \frac{\partial U}{\partial x} - i\frac{\partial U}{\partial y}$$

$$= \frac{(\ cos\ x - cosh\ y).(\cos\ x - sin\ x) + (\cos\ x + sin\ x - e^{-y}).(sin\ x)}{2(cos\ x - cosh\ y)^2}$$

$$- i\frac{(cos\ x - cosh\ y).e^{-y} + (\cos\ x + sin\ x - e^{-y}).(sinh\ y)}{2(cos\ x - cosh\ y)^2}$$

By Milne-Thomson's method, $F'(z)$ is expressed in terms of z by replacing x by z and y by 0.

Hence

$$F'(z) = \frac{(cos\ z - 1).(cos\ z - sin\ z) + (cos\ z + sin\ z - 1).sin\ z}{2(cos\ z - 1)^2} - i\frac{(cos\ z - 1)}{2(cos\ z - 1)^2}$$

$$= (1+i)\frac{1}{2(1 - cos\ z)} = (1+i)\frac{1}{4}cosec^2\frac{z}{2}$$

Integrating,

$$F(z) = -\frac{(1+i)}{2}cot\ \frac{z}{2} + C$$

$$i.e., (1+i)f(z) = -\frac{(1+i)}{2}cot\ \frac{z}{2} + C$$

$$f(z) = -\frac{1}{2}cot\ \frac{z}{2} + c$$

Given $f\left(\frac{\pi}{2}\right) = 0$, then

$$f\left(\frac{\pi}{2}\right) = -\frac{1}{2}cot\ \frac{\pi}{4} + c$$

$$0 = -\frac{1}{2} + c$$

$$c = \frac{1}{2}$$

Hence

$$f(z) = \frac{1}{2}\left(1 - cot\ \frac{z}{2}\right)$$

Unit-II
INTEGRATION IN THE COMPLEX PLANE

2.1. Introduction:

In our usual study of the subject of calculus, we practically very often relate the definite integral most closely to differentiation, thus forgetting the real meaning of the definite integral. This is because most of the definite integrals relating to functions of a real variable can be evaluated by knowing their antiderivatives. But in the case of functions of a complex variable, certain integrals of analytic functions can be evaluated without knowing the antiderivatives. For this purpose, it is very necessary that the definition of the definite integral must be thoroughly understood. Because of the two dimensional character of complex variables, first we shall develop the idea of a definite integral in the real cartesian plane.

2.2 Line Integrals:

Let $f(x, y)$ be a real function of the real variables x and y, continuous in both x and y and C be a continuous curve of finite length with initial point $A(x_0,\ y_0)$ and terminal point $B(x_n,\ y_n)$. $f(x, y)$ has no relation to the equation of C and is merely a function defined at every point in some region of the xy-plane containing the curve C. Further, the curve C is such that it is cut by a line parallel to either coordinate axis in only one point.

We divide the arc AB into n arcs $\Delta s_1, \Delta s_2, \dots\dots \Delta s_n$ whose projections on the x-axis are $\Delta x_1, \Delta x_2, \dots\dots \Delta x_n$ and whose projections on the y-axis are $\Delta y_1, \Delta y_2, \dots\dots \Delta y_n$ respectively as shown in fig. 1. Let Δs_k be a typical arc and (x_k, y_k) be the coordinates of an arbitrary point in it.

We find the value of the function $f(x, y)$ ate each of the points (x_k, y_k) and form the products $f(x_k, y_k).\Delta s_k, f(x_k, y_k).\Delta x_k$ and $f(x_k, y_k).\Delta y_k$. On assuming these products over all the subdivisions of the arc AB, we have the sums

$$\sum_{k=1}^{n} f(x_k, y_k).\Delta s_k\, , \sum_{k=1}^{n} f(x_k, y_k).\Delta x_k \ and \ \sum_{k=1}^{n} f(x_k, y_k).\Delta y_k$$

The limiting values of these sums as n becomes infinite in such a way that each Δs_k approaches zero are known as ***line integrals***.

They are written as

$$\int_C f(x,y)ds\ ,\quad \int_C f(x,y)dx \quad \text{and} \int_C f(x,y)dy \text{ respectively.}$$

C is called the field of integration.

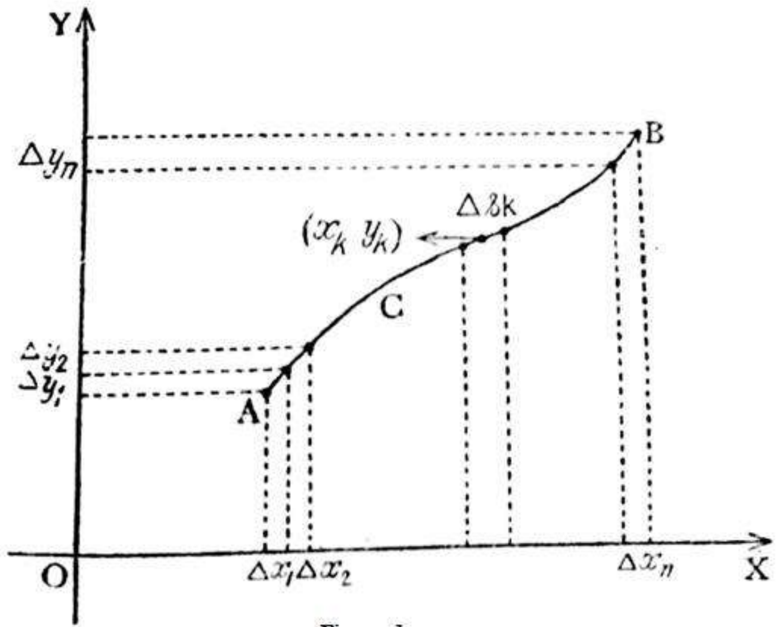


Figure 1

From these definitions, it is clear that the following familiar properties of ordinary definite integrals are true for the integrals, provided that the curve joining A and B remains the same:

$$\int_A^B c\ F\ dt = c\int_A^B F\ \ dt, c \text{ being any constant .}$$

$$\int_A^B F\ \ dt = -\int_B^A F\ \ dt$$

$$\int_A^B (F_1 \pm F_2)\ dt = \int_A^B F_1\ \ dt \pm \int_A^B F_2\ \ dt$$

$$\int_A^B F\ \ dt = \int_A^C F\ \ dt + \int_C^B F\ \ dt$$

The ordinary real definite integral like $\int_a^b f(x)\ \ dx$ can be regarded as a line integral in which the integrand is a function of x alone and the curve C is the x-axis. Also, the evaluation of line integrals can be reduced to the evaluation of ordinary definite integrals, as shown in the following examples.

EX.1. Evaluate

$$\int_{(0,\ 0)}^{(1,\ 2)} xy\ ds$$

along the three different paths shown in the figure below.

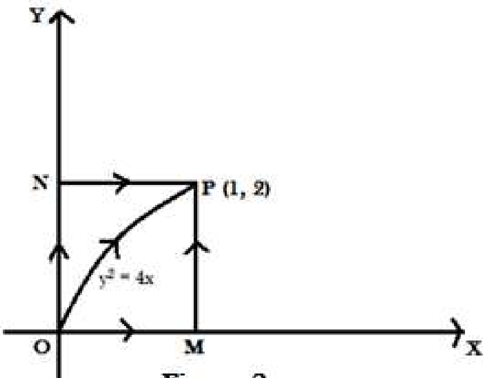


Figure 2

Solution: (i) Let us first integrate from $(0,\ 0)$ to $(1, 2)$ along the parabola $y^2 = 4x$.

Differentiating the curve's equation with respect to x, we get

$$2y\frac{dy}{dx} = 4\ i.e., \frac{dy}{dx} = \frac{2}{y}$$

We know that

$$\left(\frac{ds}{dx}\right)^2 = 1 + \left(\frac{dy}{dx}\right)^2$$

$$i.e., \left(\frac{ds}{dx}\right)^2 = 1 + \frac{4}{y^2} = 1 + \frac{4}{4x} = \frac{x+1}{x}$$

$$\therefore \frac{ds}{dx} = \frac{\sqrt{x+1}}{\sqrt{x}}\ or\ ds = \frac{\sqrt{x+1}}{\sqrt{x}}\ dx$$

Also $y = 2\sqrt{x}$ along the path OP. We can now express the line integral completely in terms of x.

$$\int_{(0,\ 0)}^{(1,\ 2)} xy\ ds = \int_{x=0}^{x=1} x\,.2\sqrt{x}.\frac{\sqrt{x+1}}{\sqrt{x}}\ dx$$

$$= 2\int_{0}^{1} x\sqrt{x+1}\,dx$$

Put $x + 1 = u^2$then $dx = 2u\ du$

When $x = 1, u = \sqrt{2}$ and when $x = 0, u = 1$.

$$\therefore \int_{(0,\ 0)}^{(1,\ 2)} xy\ ds = 2\int_{1}^{\sqrt{2}} (u^2-1)\,u.2u\,du$$

$$= 2\int_{1}^{\sqrt{2}} (u^4-u^2)\,du$$

$$= \frac{8}{15}\left(\sqrt{2}+1\right) \qquad (1)$$

(ii) Let us now evaluate the line integral along the rectilinear path OMP. We do the integration in two stages.

$$\int_{O(0,\ 0)}^{P(1,\ 2)} xy\ ds = \int_{O(0,\ 0)}^{M(1,\ 0)} xy\ ds + \int_{M(1,\ 0)}^{P(1,\ 2)} xy\ ds$$

Along the first path OM, $ds = dx$ and $y = 0$.

$$\therefore \int_{O(0,\ 0)}^{M(1,\ 0)} xy\ ds = \int_{x=0}^{x=1} x\,.\,0\,dx = 0$$

Along the path MP, $ds = dy$ and $x = 1$.

$$\therefore \int_{M(1,\ 0)}^{P(1,\ 2)} xy\ ds = \int_{y=0}^{y=2} 1\,.\,y.dy = \left[\frac{y^2}{2}\right]_0^2 = 2$$

$$Hence \int_{O(0,\ 0)}^{P(1,\ 2)} xy\ ds = 0 + 2 = 2 \qquad (2)$$

(iii) Let us now evaluate the line integral along ONP.

$$\int_{O(0,\ 0)}^{P(1,\ 2)} xy\ ds = \int_{O(0,\ 0)}^{N(0,\ 2)} xy\ ds + \int_{N(0,\ 2)}^{P(1,\ 2)} xy\ ds$$

Along the first path ON, $x = 0$ and $ds = dy$. So the value of the first integral will be equal to zero.

Along $NP, y = 2$ and $ds = dx$.

Hence the second integral

$$= \int_{x=0}^{x=1} x\,.\,2\,dx = 1$$

$$\therefore \int_{O(0,\ 0)}^{P(1,\ 2)} xy\ ds = 0 + 1 = 1 \qquad (3)$$

From results (1), (2) and (3), we find that, in general, a line integral depends not only on the terminal points of integration but also upon the particular path which joins them.

Note: If the paths were traversed in the opposite sense i.e., from (1, 2) to (0, 0), the values of the integral will be

$$-\frac{8}{15}\left(\sqrt{2}+1\right), -2 \text{ and } -1 \text{ respectively.}$$

EX.2. Find the value of

$$\int_{(0,\ 0)}^{(1,\ 3)} [x^2 y dx + (x^2 - y^2) dy]$$

along (i) $y = 3x^2$ (ii) $y = 3x$.

Solution: Let I denote the given line integral. It can be changed completely in terms of x.

(i) Since $y = 3x^2$ then $dy = 6x\ dx$

Substituting for y and dy in terms of x, we have

$$I = \int_{x=0}^{x=1} [3x^4 dx + (x^2 - 9x^4) 6x\ dx]$$

$$= \int_{x=0}^{x=1} (3x^4 + 6x^3 - 54x^5)\ dx$$

$$= \left[\frac{3x^5}{5} + \frac{6x^4}{4} - \frac{54x^6}{6}\right]_0^1 = -\frac{69}{10}$$

(ii) Since $y = 3x$ then $dy = 3dx$

Substituting for y and dy in terms of x, we have

$$I = \int_{x=0}^{x=1} [3x^3 dx + (x^2 - 9x^2) 3\ dx]$$

$$= \int_{x=0}^{x=1} (3x^3 - 24x^2)\ dx$$

$$= \left[\frac{3x^4}{4} - \frac{24x^3}{3}\right]_0^1 = -\frac{29}{4}$$

EX.3. Find the value of

$$\int_{(0,\ 0)}^{(1,\ 3)} [3x^2y dx + (x^3 - 3y^2)dy]$$

along (i) $y = 3x^2$ (ii) $y = 3x$.

Solution: Let I denote the given line integral. It can be changed completely in terms of x.

(i) Since $y = 3x^2$ then $dy = 6x\ dx$

Substituting for y and dy in terms of x, we have

$$I = \int_{x=0}^{x=1} [9x^4 dx + (x^3 - 27x^4)6x\ dx]$$

$$= \int_{x=0}^{x=1} (15x^4 - 162x^5)\, dx$$

$$= \left[\frac{15x^5}{5} - \frac{162x^6}{6}\right]_0^1 = -24$$

(ii) Since $y = 3x$ then $dy = 3dx$

Substituting for y and dy in terms of x, we have

$$I = \int_{x=0}^{x=1} [9x^3 dx + (x^3 - 27x^2)3\ dx]$$

$$= \int_{x=0}^{x=1} (12x^3 - 81x^2)\, dx$$

$$= \left[\frac{12x^4}{4} - \frac{81x^3}{3}\right]_0^1 = -24$$

Note: The two values of$I$ in this problem are the same. In fact, we can verfy that the value of I over any other path connecting the points (0, 0) and (1, 3) is also -24. Thus the value of this integral depends only on the end points and not upon the curve joining them. The reason for this remarkable behaviour will be learnt later on.

$\boldsymbol{EX.4.}$ Evaluate $\int_0^{1+i} (x^2 - iy)\, dz$ along the paths $(i)\ y = x \quad (ii)\ y = x^2$.

Solution: (i) Along OB whose equation is $y = x \ \Rightarrow dy = dx$ and x varies from 0 to 1.

$$\therefore \int_{0}^{1+i} (x^2 - iy)\, dz = \int_{(0,0)}^{(1,1)} (x^2 - iy)\,(dx + i\, dy)$$

$$\therefore \int_{OB} (x^2 - iy)\, dz = \int_{x=0}^{1} (x^2 - iy)\,(dx + i\, dy)$$

$$= \int_{x=0}^{1} (x^2 - ix)\,(dx + i\, dx) \qquad = (1+i) \int_{x=0}^{1} (x^2 - ix)\, dx$$

$$= (1+i)\left[\frac{x^3}{3} - i\frac{x^2}{2}\right]_0^1$$

$$= (1+i)\left(\frac{1}{3} - \frac{i}{2}\right)$$

(ii) Along the parabola whose equation is $y = x^2 \Rightarrow dy = 2x\, dx$.

$$\text{Now} \int_{0}^{1+i} (x^2 - iy)\, dz = \int_{(0,0)}^{(1,1)} (x^2 - iy)\,(dx + i\, dy)$$

$$\therefore \int_{OC} (x^2 - iy)\, dz = \int_{x=0}^{1} (x^2 - ix^2)\,(dx + i2x\, dx)$$

$$= (1-i) \int_{x=0}^{1} x^2\,(1 + 2ix) dx$$

$$= (1-i) \int_{x=0}^{1} (x^2 + 2ix^3)\, dx$$

$$= (1-i)\left[\frac{x^3}{3} + i\frac{x^4}{2}\right]_0^1$$

$$= (1-i)\left(\frac{1}{3} + \frac{i}{2}\right)$$

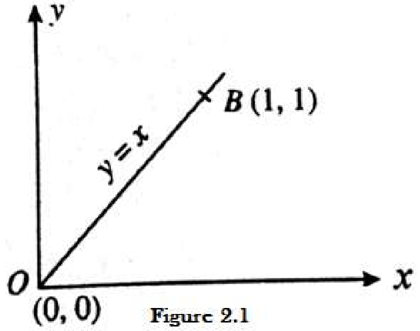


Figure 2.1

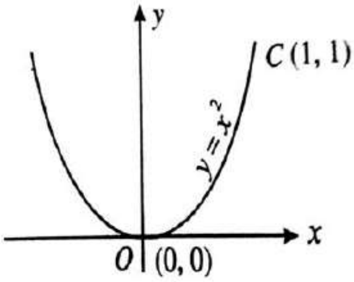


Figure 2.2

EX.5. Integrate $f(z) = x^2 + ixy$ from $A(1,1)$to B(2, 8) along

(i) The straight line AB (ii) The curve $C: x = t, y = t^3$

Solution: (i) $\int_C f(z)\,dz = \int_{(1,1)}^{(2,8)} (x^2 + ixy)\,(dx + idy)$

Along AB: Equation of AB passing through $A(1,1)$to B(2, 8) is

$$\frac{y-1}{8-1} = \frac{x-1}{2-1} \Rightarrow y = 7x - 6 \text{ and } dy = 7dx$$

$$\therefore \int_{AB} f(z)\,dz = \int_{x=1}^{2} [x^2 + ix(7x-6)](dx + 7idx)$$

$$= (7i+1)\int_{x=1}^{2} [(7i+1)x^2 - 6ix]\,dx$$

$$= (7i+1)\left[(7i+1)\frac{x^3}{3} - 3ix^2\right]_1^2$$

$$= \frac{7i+1}{3}(22i+7)$$

(ii) Along C whose parametric equations are

$$x = t, y = t^3$$

Then $dx = dt, dy = 3t^2dt$

$A(1,1) \Rightarrow t = 1$ and B(2, 8)$\Rightarrow t = 2$

$$\int_C f(z)\,dz = \int_{(1,1)}^{(2,8)} (x^2 + ixy)\,(dx + idy)$$

$$\therefore \quad \int_C f(z)\,dz = \int_{t=1}^{t=2} (t^2 + it^4)\,(dt + i3t^2dt)$$

$$= \int_1^2 (t^2 + it^4)\,(1 + i3t^2)\,dt$$

$$= \int_1^2 (t^2 + it^4 + 3it^4 - 3t^6)\,dt$$

$$= \int_1^2 [t^2 + (1+3i)t^4 - 3t^6]\,dt$$

$$= \left[\frac{t^3}{3} + (1+3i)\frac{t^5}{5} - 3\frac{t^7}{7}\right]_1^2$$

$$= \frac{1}{105}[-4818 + i1953]$$

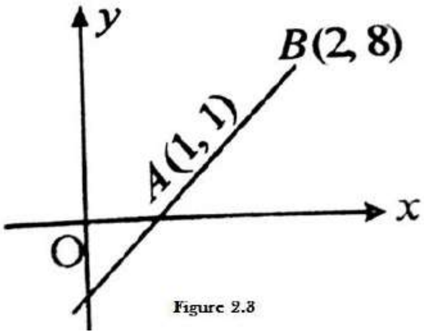


Figure 2.3

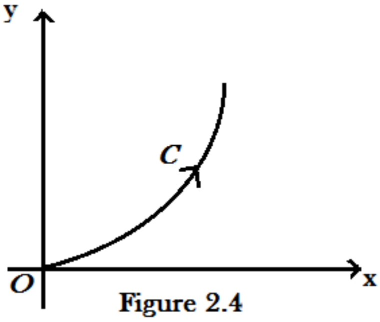


Figure 2.4

***EX*. 6.** Evaluate $\int_C (x-2y)dx + (y^2 - x^2)dy$ where C is the boundary of the first quadrant of the circle $x^2 + y^2 = 4$.

Solution: Parametric equations of the circle are

$x = 2\cos\theta$ and $y = 2\sin\theta,$ where $0 \le \theta \le 2\pi$

Then $dx = -2\sin\theta\, d\theta$ and $dy = 2\cos\theta\, d\theta$

$$\therefore \int_C (x-2y)dx + (y^2 - x^2)dy$$

$$= \int_0^{\pi/2} [(2\cos\theta - 4\sin\theta)(-2\sin\theta\, d\theta) + (4\sin^2\theta - 4\cos^2\theta)(2\cos\theta\, d\theta)]$$

$$= -4\left[\frac{1}{2}\int_0^{\pi/2} \sin 2\theta\, d\theta - \int_0^{\pi/2} (1-\cos 2\theta)\, d\theta - 2\int_0^{\pi/2} \sin^2\theta\cos\theta\, d\theta + 2\int_0^{\pi/2} \frac{1}{4}(\cos 3\theta + 3\cos\theta)\, d\theta\right]$$

$$= 2\pi - \frac{14}{3}$$

EX. 7. *Evaluate* $\int_0^{2+i} z^2\, dz$ *along*

(i) the real axis to 2 and then vertically to $(2+i)$.

(ii) the imaginary axis to i and then horizontally to $(2+i)$.

Solution: $I = \int_0^{2+i} z^2\, dz = \int_{OA} z^2\, dz + \int_{AB} z^2\, dz$ (1)

We have $z = x + iy$ then $dz = dx + idy$

(i) Along $OA, y = 0$

$\therefore \quad z = x + iy = x$ and $dz = dx$

$$\therefore \quad \int_{OA} z^2\, dz = \int_0^2 x^2\, dx = \left[\frac{x^3}{3}\right]_0^2 = \frac{8}{3} \qquad (2)$$

(ii) Along $AB, x = 2$

$\therefore \quad z = x + iy = 2 + iy$ and $dz = idy$

Also $z^2 = (2 + iy)^2 = 4 + 4iy - y^2$

and y varies from 0 to 1.

$$\therefore \quad \int_{OA} z^2\, dz = i\int_0^1 (4 + 4iy - y^2)\, dy$$

$$= i\left[4y + 2iy^2 - \frac{y^3}{3}\right]_0^1 = -2 + \frac{11i}{3} \qquad (3)$$

From (1), (2) and (3), we have

$$I = \frac{8}{3} - 2 + \frac{11i}{3} = \frac{1}{3}(2 + 11i)$$

(ii) Along $OP, x = 0$

$\therefore \quad z = x + iy = iy$ and $dz = idy$

Also $z^2 = (iy)^2 = -y^2$ and y varies from 0 to 1.

$$\therefore \quad \int_{OP} z^2\, dz = \int_0^1 (-y^2)\, idy = -i\left[\frac{y^3}{3}\right]_0^1 = -\frac{i}{3} \qquad (4)$$

Along $PQ, y = 1$ then $z = x + i$ and $dz = dx$ and x varies from 0 to 2.

$\therefore \quad z^2 = (x + i)^2 = x^2 - 1 + 2ix$

$$\therefore \quad \int_{PQ} z^2\, dz = \int_0^2 (x^2 - 1 + 2ix)\, dx = \left[\frac{x^3}{3} - x + ix^2\right]_0^2 = \frac{2}{3} + 4i \qquad (5)$$

From (1), (4) and (5), we have

$$I = -\frac{i}{3} + \frac{2}{3} + 4i = \frac{2 + 11i}{3}$$

Note: From the, we conclude that the complex line integral is independent of the path of integration.

***EX*. 8.** Evaluate $(a)\ \int_{(0,0)}^{(1,3)} 3x^2y\,dx + (x^3 - 3y^2)dy \quad (b)\ \int_{(0,0)}^{(1,3)} x^2y\,dx + (x^2 - y^2)dy$

along the curve $(i)\ y = 3x \quad (ii)\ y = 3x^2$.

Solution: Let I denote the given integral. it can be changed completely in terms of x.

(i) Since $y = 3x, dy = 3\,dx$

Substituting for y and dy in terms of x, we have

$$(a)\quad I = \int_0^1 9x^3\,dx + \int_0^1 (x^3 - 27x^2)(3dx)$$

$$= \int_0^1 (12x^3 - 81x^2)dx = \left[12\frac{x^4}{4} - 81\frac{x^3}{3}\right]_0^1 = -24$$

$$(b)\quad I = \int_0^1 3x^3\,dx + \int_0^1 (x^2 - 9x^2)(3dx)$$

$$= \int_0^1 (3x^3 - 24x^2)dx = \left[3\frac{x^4}{4} - 24\frac{x^3}{3}\right]_0^1 = -\frac{29}{4}$$

(ii) Since $y = 3x^2, dy = 6x\,dx$

Substituting for y and dy in terms of x, we have

$$(a)\quad I = \int_0^1 9x^4\,dx + \int_0^1 (x^3 - 27x^4)(6x\,dx)$$

$$= \int_0^1 (15x^4 - 162x^5)dx = \left[15\frac{x^5}{5} - 162\frac{x^6}{6}\right]_0^1 = -24$$

$$(b)\quad I = \int_0^1 3x^4\,dx + \int_0^1 (x^2 - 9x^4)(6x\,dx)$$

$$= \int_0^1 (6x^3 + 3x^4 - 54x^5)dx = \left[6\frac{x^4}{4} + 3\frac{x^5}{5} - 54\frac{x^6}{6}\right]_0^1 = \frac{3}{2} + \frac{3}{5} - 9 = -\frac{69}{10}$$

Note: In the above problem (a), we find that the two values of I are the same. In fact, we can also see that the value of I over any other path joining the points $(0,0)$ and $(1,3)$ is -24.

Thus the value of this integral depends only on the end points and not upon the curve connecting them.

$\boldsymbol{EX.9.}$ Evaluate $\int_{(0,0)}^{(1,1)} (3x^2 + 4xy + ix^2)\, dz$ along $y = x^2$.

Solution: Let $z = x + iy$ so that $dz = dx + idy$

$$\text{Now} \int_{(0,0)}^{(1,1)} (3x^2 + 4xy + ix^2)\, dz = \int_{(0,0)}^{(1,1)} (3x^2 + 4xy + ix^2)\,(dx + idy) \quad (1)$$

Along $y = x^2, dy = 2x\, dx$.

On putting the values of y and dy, (1) becomes

$$\int_{(0,0)}^{(1,1)} (3x^2 + 4xy + ix^2)\ dz = \int_0^1 (3x^2 + 4x^3 + ix^2)\,(dx + i2x\, dx)$$

$$= \int_0^1 [(3+i)x^2 + 4x^3]\,(1 + i2x)dx$$

$$= \int_0^1 [(3+i)x^2 + 2(3i+1)x^3 + i8x^4]\, dx$$

$$= \left[(3+i)\frac{x^3}{3} + 2(3i+1)\frac{x^4}{4} + i8\frac{x^5}{5}\right]_0^1$$

$$= \frac{3}{2} + i\frac{103}{30}$$

$\boldsymbol{EX.10.}$ Evaluate $\int_C (y^2 + 2xy)dx + (x^2 - 2xy)d\,y$

where C is the region bounded by $y = x^2$ and $x = y^2$.

Solution: Given curves are $y = x^2$ (1)

and $x = y^2$ (2)

The two curves (1) and (2) intersect at the points $(0,0)$ and $(1,1)$.

The positive direction in traversing C is as shown in figure.

Along $y = x^2, dy = 2x\, dx$, the line integral is

$$= \int_{x=0}^1 [(x^4 + 2x^3)dx + (x^2 - 2x^3)]\, 2x\, dx$$

$$= \int_0^1 (4x^3 - 3x^4)dx = \left[4\frac{x^4}{4} - 3\frac{x^5}{5}\right]_0^1$$

$$= 1 - \frac{3}{5} = \frac{2}{5} \qquad (3)$$

Along $x = y^2, dx = 2y\, dy$, the line integral is

$$= \int_{y=1}^{0} [(y^2 + 2y^3)2y\, dy + (y^4 - 2y^3)]\ dy$$

$$= \int_1^0 (2y^3 + 5y^4 - 2y^3)\, dy$$

$$= [y^5]_1^0 = -1 \qquad (4)$$

Hence, the line integral over $C = \frac{2}{5} - 1 = -\frac{3}{5}$ [adding (3) and (4)].

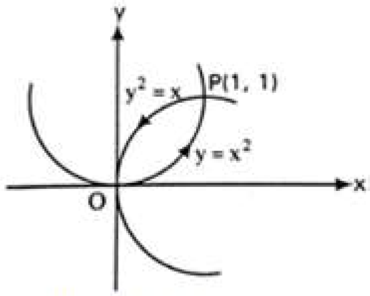


Figure 2.5

***EX*. 11.** Evaluate $\int_0^{3+i} z^2 dz$, along (i) the line $y = \frac{x}{3}$ (ii) parabola $x = 3y^2$.

Solution: Let I denote the given integral. Then we have

$$I = \int_0^{3+i} z^2 dz = \int_{(0,0)}^{(3,1)} (x + iy)^2\, (dx + idy)$$

$$= \int_{(0,0)}^{(3,1)} (x^2 - y^2 + i2xy)\, (dx + idy)$$

$$= \int_{(0,0)}^{(3,1)} [(x^2 - y^2)dx - 2xydy] + i\int_{(0,0)}^{(3,1)} [2xydx + (x^2 - y^2)dy]$$

(i) I can be changed completely in terms of x.

Since $y = \frac{x}{3}, dy = \frac{1}{3}dx$

Substituting for y and dy, we have

$$I = \int_{x=0}^{3} \left[\left(x^2 - \frac{x^2}{9}\right) dx - \frac{2}{3}x^2\left(\frac{1}{3}dx\right)\right] + i \int_{x=0}^{3} \left[\frac{2}{3}x^2 dx + \left(x^2 - \frac{x^2}{9}\right)\frac{1}{3}dx\right]$$

$$= \int_{x=0}^{3} \frac{6}{9}x^2 dx + i \int_{x=0}^{3} \frac{26}{27}x^2\, dx = \frac{6}{3}\left[\frac{x^3}{3}\right]_0^3 + i\frac{26}{27}\left[\frac{x^3}{3}\right]_0^3$$

$$= 6 + \frac{26}{3}i$$

(ii) I can be changed completely in terms of x.

Since $y = \frac{x}{3}, dy = \frac{1}{3}dx$

Substituting for y and dy, we have

$$I = \int_{y=0}^{1} [(9y^2 - y^2)6y\, dy - 6y^3 dy] + i \int_{y=0}^{1} [6y^2(6ydy) + (9y^2 - y^2)]$$

$$= \int_{y=0}^{1} 42y^3 dy + i \int_{y=0}^{1} (36y^3 + 8y^2)dy$$

$$= \left[42\frac{y^4}{4} + i\left(36\frac{y^4}{4} + 8\frac{y^3}{3}\right)\right]_0^1 \quad = \frac{21}{2} + i\frac{35}{3}$$

2.3 Properties of Line Integrals:

Theorem 1. Let$P$ and Q be two functions of x and y, such that $P, Q, \frac{\partial P}{\partial y}$ and $\frac{\partial Q}{\partial x}$ are continuous and single valued at every point of a simply connected region R. The necessary and sufficient condition that $\int_C (P\, dx + Q\, dy) = 0$ around every closed curve C drawn in R is that $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$ at all points in R.

Proof: First let us prove the **sufficiency** of the condition.

Suppose $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$ identically in R.

Let R_1 be the subregion of R bounded by the particular closed curve C_1.

Applying Green's theorem to R_1,

$$\int_{C_1} (P\, dx + Q\, dy) = \iint_{R_1} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\, dy = 0$$

Since $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$ at all points in R and so at all points in R_1 also.

We shall now prove the **necessity** of the condition.

Suppose $\int_C (P\, dx + Q\, dy) = 0$ around every closed curve C in R.

Suppose that $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y} > 0$ at some point (x_0, y_0) of R.

Since $\frac{\partial P}{\partial y}$ and $\frac{\partial Q}{\partial x}$ are continuous functions of x and y, $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$ is also a continuous function of x and y.

Hence there must be a region S about (x_0, y_0) in which

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$$

has the same sign as at (x_0, y_0).

i.e., in the region S,

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} > 0.$$

Let C_1 be the boundary of S. Applying Green's theorem to S, we have

$$\int_{C_1} (P\,dx + Q\,dy) = \iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

and this is >0, as the integrand is positive.

Hence we get that

$$\int_{C_1} (P\,dx + Q\,dy) > 0.$$

But this is against the hypothesis that $\int (P\,dx + Q\,dy) = 0$ around every closed curve in R.

Therefore $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$ cannot be >0 at (x_0, y_0).

Similarly we can show that $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$ cannot be negative at (x_0, y_0).

Therefore $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$ must be $= 0$ at (x_0, y_0).

But (x_0, y_0) is any point in R.

Therefore $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$ must be $\equiv 0$ at (x_0, y_0).

Or $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$ at every point in R.

Theorem 2. Let P and Q satisfy the conditions of theorem 1. The necessary and sufficient condition that

$$\int_A^B (P\,dx + Q\,dy)$$

be independent of the path connecting A and B is that $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$ at all points in R.

Proof: First let us prove the sufficiency of the condition.

Suppose $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$ identically in R.

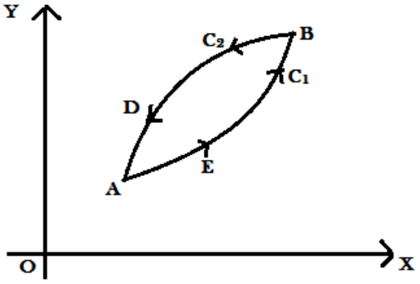


Figure 14

Let C_1 and C_2 be any two curves connecting the two points A and B. Together they form a simple closed curve in R and so we can apply the last theorem. Therefore

$$\int_{C_1+C_2} (P\,dx + Q\,dy) = 0$$

$$i.e., \int_{C_1} (P\,dx + Q\,dy) + \int_{C_2} (P\,dx + Q\,dy) = 0$$

$$i.e., \int_A^B (P\,dx + Q\,dy) + \int_B^A (P\,dx + Q\,dy) = 0$$

Therefore

$$\int_A^B (P\,dx + Q\,dy) = -\int_B^A (P\,dx + Q\,dy) = \int_A^B (P\,dx + Q\,dy)$$

$along C_1 along C_2 along C_2$

$$i.e., \int_{C_1} (P\,dx + Q\,dy) = \int_{C_2} (P\,dx + Q\,dy)$$

i.e., the line integral taken over any two paths from A to B, has the same value. So it is independent of the path joining the point A and B i.e., it is a function of the end points alone.

We shall now prove the necessity of the condition.

$$\text{Suppose} \int_A^B (P\,dx + Q\,dy)$$

Is independent of the path from A to B.

Then for any two curves C_1 and C_2 connecting A and B we have

$$\int_{C_1} (P\,dx + Q\,dy) = \int_{C_2} (P\,dx + Q\,dy)$$

$$i.e., \int_{AEB} (P\,dx + Q\,dy) = \int_{ADB} (P\,dx + Q\,dy), \text{from figure}$$

$$= -\int_{BDA} (P\,dx + Q\,dy)$$

$$\text{Therefore } \int_{AEB} (P\,dx + Q\,dy) + \int_{BDA} (P\,dx + Q\,dy) = 0$$

$$i.e., \int_{AEBDA} (P\,dx + Q\,dy) = 0$$

i.e., the line integral around any closed path in R is zero.

Hence by theorem 1, $\dfrac{\partial Q}{\partial x} = \dfrac{\partial P}{\partial y}$.

EX.12. Show that

$$\int_{(0,\ 1)}^{(1,\ 2)} [(x^2 + y^2)dx + 2xy\,dy]$$

is independent of the path and determine its value.

Solution: Here $P = x^2 + y^2, Q = 2xy$

Then $\frac{\partial P}{\partial y} = 2y$ and $\frac{\partial Q}{\partial x} = 2y$

Clearly $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$ and they are also continuous.

Hence the line integral is independent of the path.

We can choose any path joining (0, 1) and (1, 2). For instance, let us take the line joining (0, 1) and (1, 1) and then the line joining (1, 1) and (1, 2).

Along the line joining (0, 1) and (1, 1), we have $y = 1$ and $dy = 0$. x varies from 0 to 1.

Along the line joining (1, 1) and (1, 2), we have $x = 1$ and $dx = 0$. y varies from 1 to 2.

Hence the required integral is

$$= \int_0^1 (x^2 + 1)dx + \int_1^2 2y\,dy$$

$$= \left[\frac{x^3}{3} + x\right]_0^1 + 2\left[\frac{y^2}{2}\right]_1^2 = \frac{13}{3}.$$

EX.13. Show that the line integral

$$\int_C \left[\frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy\right]$$

evaluated along a square 2 units on the side with center at the origin has value 2π. Give the reason for failure of this integral to vanish along this closed path.

Solution: The square $ABCD$ is formed by the lines $x = \pm 1, y = \pm 1$.

The direction in which the square C is traversed is shown in the figure 15.

Along AB, $y = -1$, $dy = 0$ and x varies from -1 to 1.

So the line integral along AB is

$$\int_{-1}^{1} \frac{1}{x^2+1}dx = [tan^{-1}x]_{-1}^{1}$$

$$= tan^{-1}1 - tan^{-1}(-1)$$

$$= \frac{\pi}{4} - \left(-\frac{\pi}{4}\right) = \frac{\pi}{2}$$

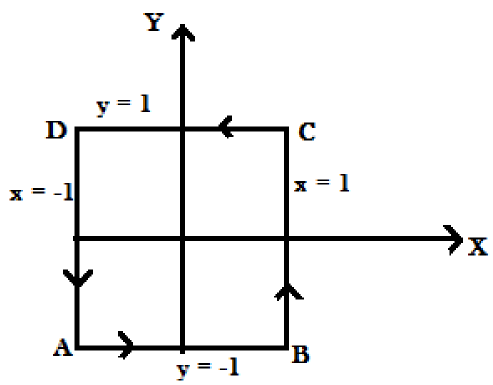


Figure 15

Along BC, $x = 1$, $dx = 0$ and y varies from -1 to 1.

So the line integral along BC is

$$= \int_{-1}^{1} \frac{1}{1+y^2}dy = [tan^{-1}y]_{-1}^{1}$$

$$= tan^{-1}1 - tan^{-1}(-1)$$

$$= \frac{\pi}{4} - \left(-\frac{\pi}{4}\right) = \frac{\pi}{2}$$

Along CD, $y = 1$, $dy = 0$ and x varies from 1 to -1.

So the line integral along CD is

$$= \int_{1}^{-1} \frac{-1}{x^2+1}\,dx = \int_{-1}^{1} \frac{1}{x^2+1}\,dx = \frac{\pi}{2}$$

Along DA,$x = -1$, $d\,x = 0$ and y varies from 1 to -1.

So the line integral along DA is

$$= \int_{1}^{-1} \frac{-1}{1+y^2}\,dy == \int_{-1}^{1} \frac{1}{1+y^2}\,dy$$

Adding the above four results, the value of the given line integral along $C = 4 \times \frac{\pi}{2} = 2\pi$.

Now $P = \dfrac{-y}{x^2+y^2}, Q = \dfrac{x}{x^2+y^2}$, then we have

$$\frac{\partial P}{\partial y} = -\left[\frac{(x^2+y^2).1 - y.2y}{(x^2+y^2)^2}\right] = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\frac{\partial Q}{\partial x} = \frac{(x^2+y^2).1 - x.2x}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

We find that $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$. Also $P, Q, \frac{\partial P}{\partial y}$ and $\frac{\partial Q}{\partial x}$ are continuous and single valued for all points of the xy-plane concept (0, 0).

Hence $\int_C (Pdx + Qdy) = 0$ around any closed curve C which does not enclose $(0, 0)$.

But here the square $ABCD$ encloses the origin and hence the line integral along $ABCD$ is not zero.

2.4 Complex Integration:

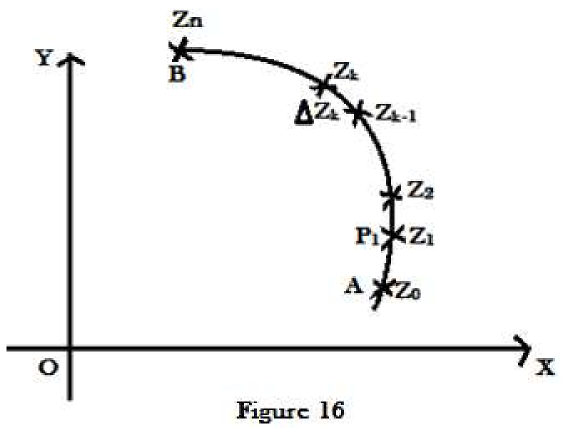


Figure 16

Let $w = f(z)$ be a continuous function of the complex variable $z = x + iy$. Let C be any curve connecting two points A and B. We divide C into n parts at the points

$$A = z_0, z_1, z_2, \ldots\ldots, z_n = B.$$

Let $\Delta z_k = z_k - z_{k-1}$

and $\boldsymbol{\zeta_k}$ be an arbitrary point in the arc $z_{k-1}z_k$. Then the limit of the sum

$$\sum_{k=1}^{n} f(\boldsymbol{\zeta_k})\ \Delta z_k \text{as } n \to \infty$$

In such a way that the length of every chord Δz_k approaches zero, is called the line integral of $f(z)$ along C. This is written as

$$\int_C f(z)\ dz = \lim_{n\to\infty} \sum_{k=1}^{n} f(\boldsymbol{\zeta_k})\ \Delta z_k$$

It can be noted that this definition differs from the definition of a real line integral in that it is based on the directed chord Δz_k tending to zero and not on the arc Δs_k tending to zero. Also the real definite integral can be interpreted as an area. It has also physical interpretation. But a line integral in the complex plane has no corresponding interpretation. However, the theory of integration in the complex plane has remarkable applications in engineering, physics etc.

We can express a complex line integral in terms of real line integral. Taking

$$w = f(z) = u(x, y) + iv(x, y),$$

and noting that $dz = dx + idy$, we have

$$\int_C f(z)\ dz = \int_C (u + iv)\ (dx + idy)$$

$$= \int_C (udx - vdy) + i\int_C (vdx + udy)$$

The two integrals on the right side are clearly line integrals of real functions.

Since a complex line integral can be expressed in terms of real line integrals, the following familiar properties are true for complex line integral also, provided the same path of integration is used in each integral. Thus

$$\int_A^B f(z)\, dz = -\int_B^A f(z)\, dz$$

$$\int_A^B k\, f(z)\, dz = k \int_A^B f(z)\, dz$$

$$\int_A^B [f(z) \pm g(z)] dz = \int_A^B f(z)\, dz \pm \int_A^B g(z)\, dz$$

$$\int_A^B f(z)\, dz = \int_A^C f(z)\, dz + \int_C^B f(z)\, dz$$

where A, B, C are any three points on the path of integration.

EX.14. If C is a circle of radius r and centre a, prove that

$$(a) \int_C \frac{dz}{z-a} = 2\pi i$$

$$(b) \int_C \frac{dz}{(z-a)^{n+1}} = 0, \text{where } n \text{ is an integer.}$$

Solution: Let A represent the fixed complex number a and P a variable point z on the circle. Then $AP = z - a$. Let AP make an angle θ with the real axis. Then $AP = re^{i\theta}$, as r is its length. Therefore

$$z - a = re^{i\theta}$$

This is the parametric equation to the circle C and θ varies from 0 to 2π, r being constant.

Therefore $dz = ri\, e^{i\theta} d\theta$

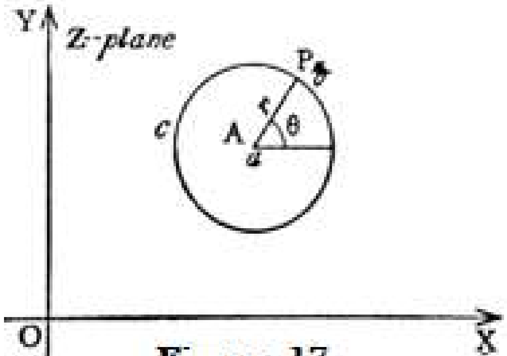


Figure 17

$$\text{(a)} \int_C \frac{dz}{z-a} = \int_0^{2\pi} \frac{ri\, e^{i\theta}}{re^{i\theta}} d\theta = \int_0^{2\pi} i\, d\theta = 2\pi i$$

$$(b)\int_C \frac{dz}{(z-a)^{n+1}} = \int_0^{2\pi} \frac{ri\,e^{i\theta}}{(r\,e^{i\theta})^{n+1}}d\theta = \int_0^{2\pi} \frac{ri\,e^{i\theta}}{r^{n+2}e^{(n+1)\theta}}d\theta$$

$$= \frac{1}{r^n}\int_0^{2\pi} e^{-in\theta}\,d\theta = \frac{i}{r^n}\int_0^{2\pi}(\cos n\theta - i\sin n\theta)\,d\theta$$

$$= \frac{i}{r^n}\left[\frac{\sin n\theta}{n} + \frac{i\cos n\theta}{n}\right]_0^{2\pi} = 0$$

These two results are important and will be of use later on.

EX. 15. Evaluate $\int_0^{2+i} z^2 dz$ along (i)the line $x = 2y$ (ii) the real axis to 2 and then vertically to $2+i$ (iii) the imaginary axis to i and then horizontally to $2+i$.

Solution:

Let $I = \int_0^{2+i} z^2 dz$

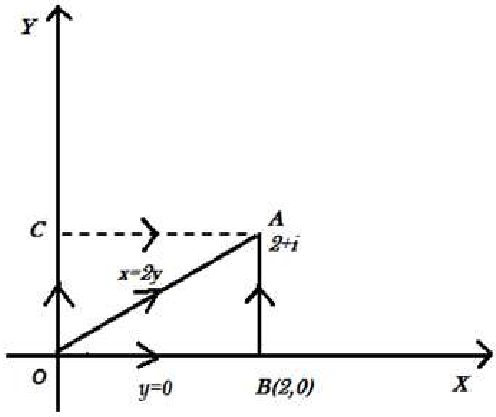


Figure 18

(i) Along OA, $x = 2y$

$$z = x + iy = 2y + iy = (2+i)y$$

Therefore

$$z^2 = (2+i)^2 y^2 = (3+4i)y^2$$

$$\text{and } dz = (2+i)dy$$

when $z = 0, y = 0$ and when $z = 2+i, y = 1$. Therefore

$$I = \int_{y=0}^{1} (3+4i)y^2 . (2+i)dy$$

$$= (3+4i)(2+i)\left[\frac{y^3}{3}\right]_0^1$$

$$= \frac{2 + 11i}{3}$$

(ii) We now evaluate I along the contour OBA as shown in figure.

$$I = \int_{OB} z^2\, dz + \int_{BA} z^2\, dz$$

Along $OB, y = 0$ then $z = x$ and $dz = dx$.

$$\int_{OB} z^2\, dz = \int_{x=0}^{2} x^2\, dx = \left[\frac{x^3}{3}\right]_0^2 = \frac{8}{3}$$

Along $BA, x = 2; z = x + iy = 2 + iy; dz = idy$

$z^2 = (2 + iy)^2 = 4 + 4iy - y^2; y$ varies from 0 to 1.

$$\int_{BA} z^2\, dz = \int_{y=0}^{1} (4 - y^2 + 4iy)\, idy$$

$$= \left[4y - \frac{y^3}{3} + 2iy^2\right]_0^1$$

$$= i\left(4 - \frac{1}{3} + 2i\right)$$

$$= -2 + \frac{11i}{3}$$

Hence $I = \frac{8}{3} - 2 + \frac{11i}{3} = \frac{2 + 11i}{3}$

(iii) We now evaluate I along the contour OCA.

$$I = \int_{OC} z^2\, dz + \int_{CA} z^2\, dz$$

Along $OC, x = 4$ then $z = iy$ and $dz = idy$.

$z^2 = i^2y^2 = -y^2, y$ varies from 0 to 1

$$\int_{OC} z^2\, dz = \int_{0}^{1} -y^2\, idy = -i\left[\frac{y^3}{3}\right]_0^1 = -\frac{i}{3}$$

Along $CA, y = 1$then $z = x + iy = x + i$ and $dz = dx$ and x varies from 0 to 2.

$z^2 = (x + i)^2 = x^2 - 1 + 2ix.$

$$\int_{CA} z^2\, dz = \int_{0}^{1} (x^2 - 1 + 2ix)\, dx = \left[\frac{x^3}{3} - x + ix^2\right]_0^2$$

$$= \frac{8}{3} - 2 + 4i = \frac{2}{3} + 4i$$

Hence $\quad I = -\frac{i}{3} + \frac{2}{3} + 4i = \frac{2 + 11i}{3}$

Note: In this problem, we find that the complex line integral is independent of the path of integration. The reason for this will be learnt later on.

EX.16. Evaluate

$$\int_{0}^{1+i} (x - y + ix^2)\, dz$$

along the line from $z = 0$ to $z = 1 + i$.

Solution:

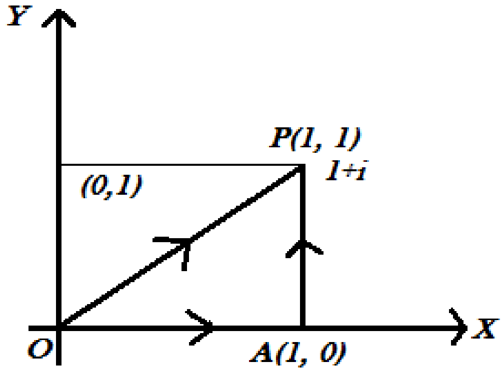


Figure 19

Let $z = x + iy$, then $dz = dx + idy$

Now $\int_{0}^{1+i} (x - y + ix^2)\, dz = \int_{0}^{1+i} (x - y + ix^2)\,(dx + idy)$

$$= \int_{OP} (x - y + ix^2)\,(dx + idy)$$

The equation of OP is $y = x$, then $dy = dx$ and x varies from 0 to 1. Therefore

$$\int_{0}^{1+i} (x - y + ix^2)\, dz = \int_{0}^{1} (ix^2)\,(dx + idx)$$

$$= \int_{0}^{1} (ix^2)\,(1 + i)dx$$

$$= (-1 + i)\int_{0}^{1} x^2\, dx$$

$$= (-1+i)\left[\frac{x^3}{3}\right]_0^1$$

$$= \frac{-1+i}{3} = -\frac{1}{3} + \frac{1}{3}i$$

EX.17. Evaluate

$$\int_0^{1+i} (x^2 - iy)\, dz$$

along the paths (a) $y = x$ (b) $y = x^2$.

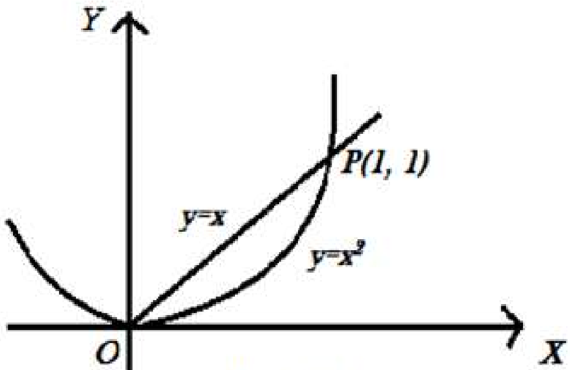


Figure 20

Solution: (a) Along $y = x, dy = dx, x$ varies from 0 to 1.

$$\int_0^{1+i} (x^2 - iy)\, dz = \int_0^{1+i} (x^2 - iy)\,(dx + idy)[since\ z = x + iy; dz = dx + idy]$$

$$= \int_0^1 (x^2 - ix)\,(dx + idx)$$

$$= \int_0^1 (x^2 - ix)\,(1+i)dx$$

$$= (1+i)\left[\frac{x^3}{3} - i\frac{x^2}{2}\right]_0^1$$

$$= \frac{5}{6} - i\frac{1}{6}$$

(b) Along $y = x^2, dy = 2x\, dx$ and x varies from 0 to 1.

$$\int_0^{1+i} (x^2 - iy)\, dz = \int_0^{1+i} (x^2 - iy)\,(dx + idy)$$

$$= \int_0^1 (x^2 - ix^2)\,(dx + i2x\, dx)$$

$$= (1 - i)\int_0^1 x^2\,(1 + 2ix\,)dx$$

$$= (1 - i)\left[\frac{x^3}{3} + i\frac{x^4}{2}\right]_0^1 = \frac{5}{6} + i\frac{1}{6}$$

EX. 18. Evaluate $\int_C z^2\,dz$ where the ends of C are $A(1,1)$ and $B(2,4)$ given that

(i) C is the curve $y = x^2$, (ii) C is the line $y = 3x - 2$.

Solution: (i) Along $y = x^2, dy = 2x\,dx$ and x varies from 1 to 2.

$$\therefore \int_C z^2\,dz = \int_{AB} (x + iy)^2\,(dx + idy)$$

$$= \int_{AB} (x^2 - y^2 + 2ixy)^2\,(dx + idy)$$

$$= \int_1^2 \{(x^2 - x^4)dx - 2x^3dx\} + i\int_1^2 \{2x(x^2)dx - (x^2 - x^4)2x\,dx\}$$

$$= -\frac{86}{3} - 6i$$

(ii) Along $y = 3x - 2, dy = 3dx$

$$\therefore \int_C z^2\,dz = \int_{AB} (x + iy)^2\,(dx + idy)$$

$$= \int_{AB} (x^2 - y^2 + 2ixy)^2\,(dx + idy)$$

$$= \int_{AB} \{x^2 - (3x - 2)^2 + 2ix(3x - 2)\}\,(dx + i3dx)$$

$$= -\frac{86}{3} - 6i$$

Note: The values of the integral along the two curves $y = x^2$ and $y = 3x - 2$ are the same which implies that $\int_C\ f(z)\,dz$ is independent of the path joining any two points. For proof refer to Cauchy's theorem.

EX.19. Show that

$$\int_C (Z+1)\,dz = 0$$

where C is the boundary of the square whose vertices are at the points $z=0, z=1, z=1+i$ and $z=i$.

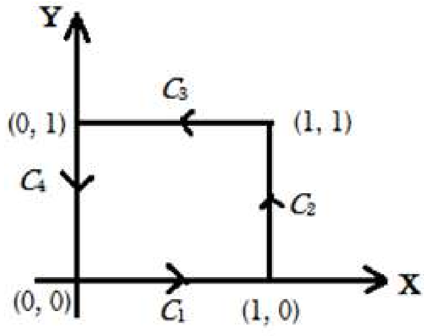


Figure 21

Solution: Let $z = x+iy$ (1)

Then $dz = dx + idy$ (2)

$$\text{Let} \int_C (Z+1)\,dz = I_1 + I_2 + I_3 + I_4 \qquad (3)$$

$$I_1 = \int_{C_1} (Z+1)\,dz = \int_{C_1} (x+iy+1)\,(dx+idy)$$

Along $C_1, y=0, dy=0, dz=dx$. Also x varies from 0 to 1.

$$\therefore I_1 = \int_0^1 (x+1)\,dx = \left[\frac{(x+1)^2}{2}\right]_0^1 = \frac{4}{2} - \frac{1}{2} = \frac{3}{2} \qquad (4)$$

Along $C_2, x=1, dx=0, dz=i\,dy$ and y varies from 0 to 1.

$$\therefore I_2 = \int_{C_2} (Z+1)\,dz = \int_{C_2} (x+iy+1)\,(dx+idy)$$

$$= \int_0^1 (1+iy+1)\,i\,dy = i\int_0^1 (2+iy)\;dy$$

$$= i\left[\frac{(2+iy)^2}{2i}\right]_0^1 = \frac{-1+4i}{2} \qquad (5)$$

Along $C_3, y=1, dy=0, dz=dx$. Also x varies from 1 to 0.

$$\therefore I_3 = \int_{C_3} (Z+1)\,dz = \int_{C_3} (x+iy+1)\,(dx+idy)$$

$$= \int_1^0 (x+i+1)\,dx = \left[\frac{x^2}{2} + (i+1)x\right]_1^0$$

$$= -\frac{3}{2} - i \qquad (6)$$

Along $C_4, x=0, dx=0, dz=i\,dy$ and y varies from 1 to 0.

$$\therefore I_4 = \int_{C_4} (Z+1)\,dz = \int_{C_4} (x+iy+1)\,(dx+idy)$$

$$= \int_1^0 (iy+1)\,i\,dy = \int_0^1 (i-y)\ dy$$

$$= \left[\frac{-y^2}{2} + iy\right]_0^1 = \frac{1}{2} - i \qquad (7)$$

Adding (4), (5), (6), (7), we get

$$I = I_1 + I_2 + I_3 + I_4$$

$$= \frac{3}{2} + \frac{-1+4i}{2} - \frac{3}{2} - i + \frac{1}{2} - i = 0$$

EX. 20. Evaluate $\int_C sin\,z\,dz$ along the line $z = 0$ to $z = i$.

Solution: Let $z = x + iy, then\ dz = dx + idy$

Given $z = 0$ to $z = i$.

$i.e., x + iy = 0 + 0i\ to\ x + iy = 0 + i$

$i.e., x = 0, y = 0\ to\ x = 0, y = 1. i.e., (0,0) to\ (0,1).$

$$Now \int_C sin\,z\,dz = \int_C sin\,(x+iy)\,(dx+idy)$$

Along $C, x = 0, dx = 0\ and\ y$ varies from 0 to 1.

$$\therefore \int_C sin\,z\,dz = \int_0^1 sin\,(iy)\,.\,i\,dy$$

$$= i\left[\frac{-cos\,(iy)}{i}\right]_0^1 = -cos\,i + 1 = 1 - cos\,i$$

EX. 21. Evaluate $\int_C e^z\,dz, C\ is\ |z| = 1.$

Solution: Put $z = e^{i\theta}$

Then $dz = i\,e^{i\theta}d\theta$

$$\therefore \int_C e^z\,dz = \int_0^{2\pi} e^{e^{i\theta}}\,i\,e^{i\theta}d\theta$$

Put $e^{i\theta} = x, then\ i\,e^{i\theta}d\theta = dx$

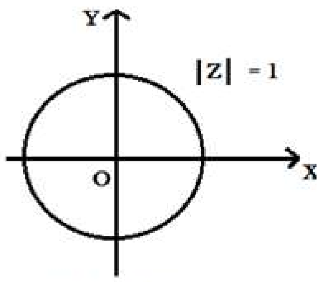


Figure 22

When $\theta = 0, x = 1$

When $\theta = 2\pi, x = 1$

Hence $\int_C e^z\, dz = \int_1^1 e^x\, dx = 0.$

EX.22. Prove that

$$\int_C \frac{dz}{z-a} = 2\pi i$$

where C is the circle $|z-a| = r$.

Solution: The equation of the circle $|z-a| = r$ can be written as

$$z - a = r\, e^{i\theta}, then\ dz = ri\, e^{i\theta} d\theta$$

Also θ varies from 0 to 2π.

$$\therefore \int_C \frac{dz}{z-a} = \int_0^{2\pi} \frac{ri\, e^{i\theta} d\theta}{r\, e^{i\theta}} = 2\pi i$$

EX.23. Evaluate

$$\int_C log\ z\ \ dz, \text{where } C \text{ is the unit circle} |z| = 1.$$

Solution: The equation of the circle $|z| = 1$ can be written as

$$z = e^{i\theta}, then\ dz = i\, e^{i\theta} d\theta$$

Also θ varies from 0 to 2π.

$$\therefore \int_C log\ z\ \ dz = \int_0^{2\pi} log\left(e^{i\theta}\right).i\, e^{i\theta} d\theta$$

$$= \int_0^{2\pi} i\theta\, .i\, e^{i\theta} d\theta \ (\because\ log\ e^x = x)$$

$$= -\int_0^{2\pi} \theta\, e^{i\theta} d\theta$$

$$= -\left[\theta\left(\frac{e^{i\theta}}{i}\right) - 1\left(\frac{e^{i\theta}}{i^2}\right)\right]_0^{2\pi} \qquad \text{[Using Bernoulli's formula]}$$

$$= -\left[\frac{2\pi e^{i2\pi}}{i} + e^{i2\pi} - 1\right]$$

$$= -\left[\frac{2\pi}{i} + 1 - 1\right] \quad (\because e^{i2\pi} = 1)$$

$$= -\frac{2\pi}{i} = 2\pi i$$

EX.24. Evaluate

$$\int_0^{2+i} (\bar{z})^2 \, dz \text{ along the line } x = 2y.$$

Solution: Let $z = x + iy$

Then $dz = dx + idy$ and $\bar{z} = x - iy$

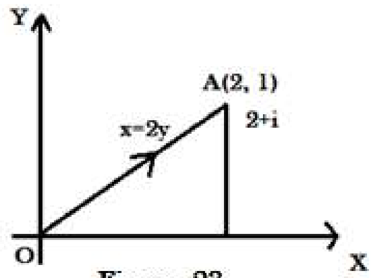


Figure 23

$(\bar{z})^2 = (x - iy)^2$

$= x^2 - y^2 - 2ixy$

$$\therefore \int_0^{2+i} (\bar{z})^2 \, dz = \int_O^A (x^2 - y^2 - 2ixy)\,(dx + idy)$$

Along $OA, x = 2y$ then $dx = 2dy$

$$\therefore \int_0^{2+i} (\bar{z})^2 \, dz = \int_0^1 (4y^2 - y^2 - 4iy^2)\,(2dy + idy)$$

$$= (2+i)\int_0^1 (3y^2 - 4iy^2)\, dy$$

$$= (2+i)\left[y^3 - 4i\frac{y^3}{3}\right]_0^1$$

$$= (2+i)\left(1 - \frac{4i}{3}\right) \qquad = \frac{5}{3}(2 - i)$$

EX. 25. Evaluate $\int_{(0,\ 0)}^{(1,\ 1)} [(x^2 + y^2)dx - 2xydy]$ along$(i) y = x \ (ii) x = y^2 (iii) y = x^2$.

Solution: (i) Along the curve$y = x, dy = dx$ and$x$varies from 0 to 1.

$$\therefore \int_{(0,\ 0)}^{(1,\ 1)} [(x^2 + y^2)dx - 2xydy] = \int_{x=0}^{1} [(x^2 + x^2)dx - 2x^2dx] = 0$$

(ii) Along the curve $x = y^2, dx = 2y\, dy$ and y varies from 0 to 1.

$$\therefore \int_{(0,\ 0)}^{(1,\ 1)} [(x^2+y^2)dx - 2xydy] = \int_{y=0}^{1} [(y^4+y^2)2ydy - 2y^3dy]$$

$$= \int_{y=0}^{1} 2y^5\, dy = \frac{1}{3}$$

(iii) Along the curve $y = x^2, dy = 2x\, dx$ and x varies from 0 to 1.

$$\therefore \int_{(0,\ 0)}^{(1,\ 1)} [(x^2+y^2)dx - 2xydy] = \int_{x=0}^{1} [(x^2+x^4)dx - 4x^4dx]$$

$$= \int_{x=0}^{1} (x^2 - 3x^4)dx = -\frac{4}{15}$$

2.5 Cauchy's Integral Theorem:

If $f(z)$ is analytic at every point of the region R bounded by a closed curve C and if $f'(z)$ is continuous throughout this closed region R, then

$$\int_C f(z)\, dz = 0.$$

Proof: Let $f(z) = u(x,y) + i\, v(x,y) = u + iv$.

Since $z = x + iy, dz = dx + idy$.

$$\text{Hence} \quad \int_C f(z)\, dz = \int_C (u+iv)\,(dx+idy)$$

$$= \int_C (udx - vdy) + i\int_C (vdx + udy) \qquad (1)$$

Since $f'(z)$ is continuous, the four partial derivatives

$$\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x} \text{ and } \frac{\partial v}{\partial y}$$

exist and are also continuous in the region R enclosed by a curve C. Hence we can apply Green's theorem, namely

$$\int_C (Pdx + Qdy) = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\, dy$$

to each of the two line integrals in the right side of (1).

$$\text{Hence} \int_C (udx - vdy) = \iint_R \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\, dy$$

$$\text{and} \int_C (vdx + udy) = \iint_R \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dx\, dy$$

Hence (1) becomes

$$\int_C f(z)\, dz = \iint_R \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\, dy + i \iint_R \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dx\, dy \qquad (2)$$

But the function $f(z)$ is analytic and so u and v satisfy the Cauchy-Riemann equations, namely

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{and} \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Hence the integrals of each of the double integrals in the right side of (2) are zero throughout the region R.

$$\text{Hence} \int_C f(z)\, dz = 0$$

Note: 1. The French Mathematician E. Goursat was the first to point out that the above Cauchy's theorem can be proved without making use of the hypothesis that $f'(z)$ is continuous. Consequently, the revised form of the theorem, usually known as the Cauchy-Goursat theorem, is stated as follows:

If a function $f(z)$ is analytic at all points interior to and on a closed contour C, then

$$\int_C f(z)\, dz = 0.$$

1. We have seen that the line integral

$$\int_C (Pdx + Qdy)$$

will be independent of the path of integration if $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$.

Now if $f(z) = u + iv$ is an analytic function,

$$\int_C f(z)\, dz = \int_C (udx - vdy) + i \int_C (vdx + udy)$$

The two integrals on the right side will be independent of the path of integration if

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \text{ respectively.}$$

But these are the well known Cauchy-Riemann equations, which are necessarily satisfied if $f(z) = u + iv$ is analytic. Hence if $f(z)$ is analytic in a simply connected region R, then the line integral of $f(z)$ is independent of the path joining any two points of R.

EX.26. If C is the boundary of the square with vertices at the points $z = 0, z = 1, z = 1 + i$ and $z = i$, show that

$$\int_C (3z + 1)\, dz = 0.$$

Solution: Given $f(z) = 3z + 1$

Since $f(z)$ is analytic everywhere (and in particular on and within the simple closed contour C).

Hence by Cauchy's theorem, it follows that

$$\int_C f(z)\, dz = 0$$

$$i.e., \int_C (3z + 1)\, dz = 0$$

EX. 27. If C is any simple closed curve, evaluate $\int_C f(z)\, dz$ if $f(z) =$

$(a) sin\, z$ $(b) cos\, 3z$ $(c) e^{2z}$ $(d) z^2 + 2$ $(e) sin\, 3z + 8z^3$

Solution: All these functions are analytic everywhere and hence in particular on and within any simple closed curve C. Hence, by Cauchy's theorem

$$\int_C f(z)\, dz = 0$$

for each one of the given functions.

2.6. Extension of Cauchy's Theorem (Cauchy's theorem for multiply connected region):

Cauchy's theorem can be applied even when the function $f(z)$ is analytic over a multiply connected region R.

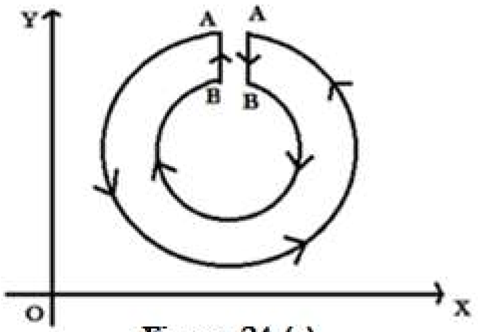


Figure 24 (a)

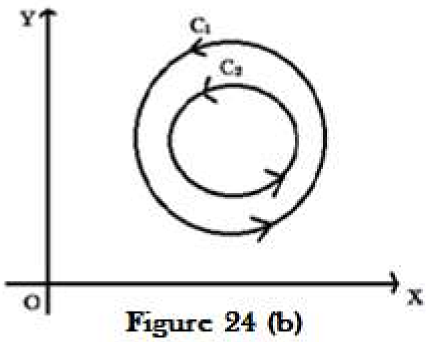


Figure 24 (b)

Let $f(z)$ be analytic in the annular region R between two closed curves C_1andC_2. By introducing the crosscut AB, the annular region is converted into a region bounded by a single curve.

We apply Cauchy's theorem to the connected contour C_1ABC_2BA and so

$$\int_C f(z)\,dz = 0$$

where the path C is a combined contour indicated by arrows: (i) along C_1 in the anticlockwise sense (ii) along AB (iii) along C_2 in the clockwise sense and (iv) along BA.

$$i.e., \int_{C_1} f(z)\,dz + \int_{AB} f(z)\,dz + \int_{C_2} f(z)\,dz + \int_{BA} f(z)\,dz = 0 \qquad (1)$$

But the integrals along AB and BA are cancel.

Therefore

$$\int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz = 0 \qquad (2)$$

provided each integral is traversed in the positive direction shown in fig. 24(a). In (2), we can reverse the direction of integration round C_2 and transpose that integral. Then, we get

$$\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz \qquad (3)$$

where each integration is now done in the anticlockwise direction as shown in fig. 24 (b). The result (3) is known as the important principle of the deformation of contours:

The line integral of a single valued analytic function $f(z)$ around any closed curve C_1 is equal to the line integral of the same function around any other closed curve C_2 into which the first can be continuously deformed without passing through a point in which $f(z)$ fails to be analytic.

If $f(z)$ is analytic in a multiply connected region bounded by the exterior contour C and the interior contours $C_1, C_2, \ldots\ldots, C_n$. The integral over the exterior contour C is equal to the sum of the integrals over the interior contours $C_1, C_2, \ldots\ldots, C_n$. It is assumed that the integration over all the contours is performed in the same direction and that $f(z)$ is analytic on all the contours.

EX.28. Consider the region $1 \leq z \leq 2$. If C is the positively oriented boundary of this region show that

$$\int_C \frac{dz}{z^2(z^2+16)} = 0$$

Solution:

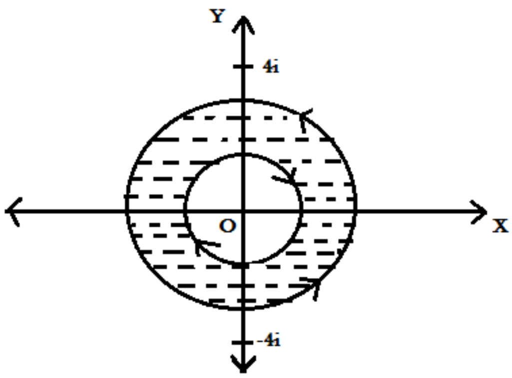


Figure 25

$|z| = 1$ and $|z| = 2$ are two circles with centre at (0, 0) and radii equal to 1 and 2 respectively.

The region $1 \leq z \leq 2$ is the dotted portion in the figure.

Let C be $|z| = 2$, the outer circle and C_1 be $|z| = 1$, the inner circle.

The positively oriented boundary of the region is obtained by tracing $|z| = 2$ in anticlockwise sense and $|z| = 1$ in clockwise sense.

The singular points of $f(z) = \dfrac{dz}{z^2(z^2+16)}$ are $z = 0$ and $z = \pm 4i$.

These three points are outside the region under consideration.

Hence, $f(z)$ is analytic on and within $|z| = 2$ but on and outside $|z| = 1$.

Hence, by the extension to Cauchy's theorem, we have

$$\int_C f(z)\, dz = 0$$

EX. 29. Evaluate $\int_C \frac{e^{2z}}{z-2}dz$ where C is $|z| = 1$.

Solution: The point $z = 2$ lies outside C.

Therefore the function $\frac{e^{2z}}{z-2}$ is analytic within and on C.

Hence by Cauchy's theorem, $\int_C \frac{e^{2z}}{z-2}dz = 0$

EX. 30. Evaluate $\oint_C (z-a)^n\, dz$, where C is a simple closed curve and the point $z = a$ is (i) inside C (ii) outside C (n is an integer).

Solution: (i) Let C: circle $z - a = r\, e^{i\theta}$, $i.e.$, a is inside C.

$$\therefore \oint_C (z-a)^n\, dz = \int_0^{2\pi} \left(r\, e^{i\theta}\right)^n . ir\, e^{i\theta} d\theta$$

$$= ir^{n+1} \int_0^{2\pi} e^{i(n+1)\theta}\, d\theta \qquad (1)$$

$$= ir^{n+1} \left[\frac{e^{i(n+1)\theta}}{i(n+1)}\right]_0^{2\pi}, \text{if } n \neq -1$$

$$= \frac{r^{n+1}}{n+1}\left[e^{i2(n+1)\pi} - 1\right], \text{if } n \neq -1$$

$$= \frac{r^{n+1}}{n+1}[cos\, 2(n+1)\pi - 1], \text{if } n \neq -1$$

$$= 0, \text{if } n \neq -1$$

If $n = -1$, then

$$\oint_C (z-a)^n\, dz = i\int_0^{2\pi} d\theta\, (from\ (1))$$

(ii) If $n = -1$ and If a is outside the circle C, then

$$\oint_C \frac{dz}{z-a} = 0 \text{ (using Cauchy's theorem)}$$

Since $\frac{1}{z-a}$ is analytic inside C.

EX.31. Verify Cauchy's theorem for the integral of z^3 taken over the boundary of the rectangle with vertices $-1, 1, 1+i, -1+i$.

Solution: The boundary of rectangle C consists of four curves c_1, c_2, c_3 and c_4. So

$$\int_C z^3\ dz = \int_{c_1} z^3\ dz + \int_{c_2} z^3\ dz + \int_{c_3} z^3\ dz + \int_{c_4} z^3\ dz \qquad (1)$$

Along c_1: $y = 0, dy = 0$ and x varies from -1 to 1.

$$\therefore \int_{c_1} z^3\ dz = \int_{-1}^{1} (x+iy)^3\ (dx + idy) = \int_{-1}^{1} x^3\ dx = 0$$

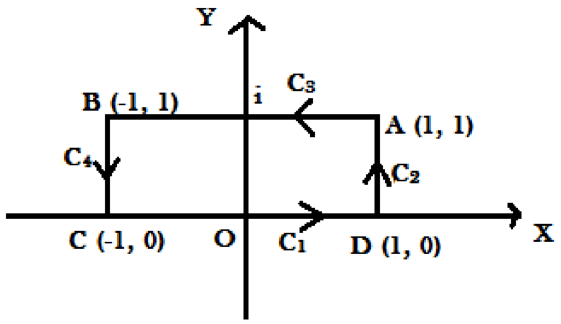


figure 26

Along c_2: $x = 1, dx = 0$ and y varies from 0 to 1.

$$\therefore \int_{c_2} z^3\ dz = \int_{0}^{1} (x+iy)^3\ (dx + idy) = \int_{0}^{1} (1+iy)^3\ idy$$

$$= i\int_{0}^{1} (1 + 3iy - 3y^2 - iy^3)\, dy = -\frac{5}{4}$$

Along c_3: $y = 1, dy = 0$ and x varies from 1 to -1.

$$\therefore \int_{c_3} z^3\ dz = \int_{1}^{-1} (x+iy)^3\, (dx + idy) = \int_{1}^{-1} (x+i)^3\ dx$$

$$= \int_{1}^{-1} (x^3 - i + 3ix^2 - 3x)\, dx = 0$$

Along c_4: $x = -1, dx = 0$ and y varies from 1 to 0.

$$\therefore \int_{c_4} z^3\ dz = \int_{1}^{0} (x+iy)^3\ (dx + idy) = \int_{1}^{0} (-1+iy)^3\ idy$$

$$= i\int_0^1 (-1 + 3iy + 3y^2 - iy^3)\, dy = \frac{5}{4}$$

Substituting the above four values in (1), we get

$$\int_C z^3\ dz = 0 - \frac{5}{4} + 0 + \frac{5}{4} = 0$$

Hence the theorem is verified.

EX.32. Verify Cauchy's theorem for the function $f(z) = 3z^2 + iz - 4$ if C is the square with vertices at $1 \pm i$ and $-1 \pm i$.

Solution:

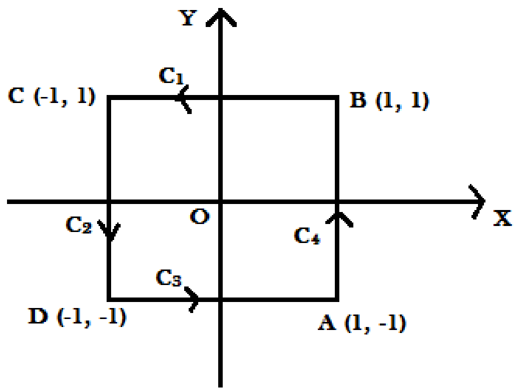


Figure 27

The boundary of square C consists of four curves c_1, c_2, c_3 and c_4. So

$$\int_C f(z)\ dz = \int_{c_1} f(z)\ dz + \int_{c_2} f(z)\ dz + \int_{c_3} f(z)\ dz + \int_{c_4} f(z)\ dz \qquad (1)$$

Along c_1: $y = 1, dy = 0$ and x varies from 1 to -1.

$$\therefore \int_{c_1} f(z)\ dz = \int_1^{-1} [3(x+iy)^2 + i(x+iy) - 4]\,(dx + idy)$$

$$= \int_1^{-1} [3(x+i)^2 + i(x+i) - 4]\,dx = 14$$

Along c_2: $x = -1, dx = 0$ and y varies from 1 to -1.

$$\therefore \int_{c_2} f(z)\ dz = \int_1^{-1} [3(x+iy)^2 + i(x+iy) - 4]\,(dx + idy)$$

$$= \int_{1}^{-1} [3(-1+iy)^2 + i(-1+iy) - 4]\ idy = -2+4i$$

Along c_3: $y = -1, dy = 0$ and x varies from -1 to 1.

$$\therefore \int_{c_3} f(z)\ dz = \int_{-1}^{1} [3(x+iy)^2 + i(x+iy) - 4]\,(dx + idy)$$

$$= \int_{1}^{-1} [3(x-i)^2 + i(x-i) - 4]\,dx = -10$$

Along c_4: $x = 1, dx = 0$ and y varies from -1 to 1.

$$\therefore \int_{c_4} f(z)\ dz = \int_{-1}^{1} [3(x+iy)^2 + i(x+iy) - 4]\,(dx + idy)$$

$$= \int_{-1}^{1} [3(1+iy)^2 + i(1+iy) - 4]\ idy = -2-4i$$

Substituting the above four values in (1), we get

$$\int_{C} z^3\ dz = 14 - 2 + 4i - 10 - 2 - 4i = 0$$

Hence the theorem is verified.

EX. 33. Show that $\int_C (z+1)dz = 0$ where C is the boundary of the square whose vertices at the points $z = 0, z = 1, z = 1+i, z = i$.

Solution:

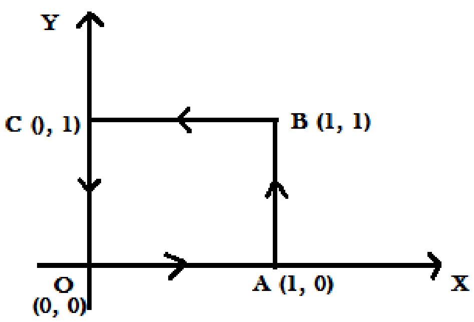


Figure 28

The boundary of square C consists of four curves OA, AB, BC and CO. So

$$\int_C (z+1)\ dz = \int_{OA} (z+1)\ dz + \int_{AB} (z+1)\ dz + \int_{BC} (z+1)\ dz + \int_{CO} (z+1)\ dz \qquad (1)$$

Along $OA, y = 0$ then $dy = 0, z = x, dz = dx$ and x varies from 0 to 1.

$$\therefore \int_{OA} (z+1)\ dz = \int_0^1 (x+1)\, dx = \frac{3}{2}$$

Along $AB, x = 1$ then $dx = 0, z = 1 + iy, dz = idy$ and y varies from 0 to 1.

$$\therefore \int_{AB} (z+1)\ dz = \int_0^1 (1+iy+1)\, idy == i\int_0^1 (2+iy)\, dy = 2i - \frac{1}{2}$$

Along $BC,\ y = 1$ then $dy = 0, z = x + i, dz = dx$ and x varies from 1 to 0.

$$\therefore \int_{BC} (z+1)\ dz = \int_1^0 (x+i+1)\, dx = -\left(\frac{3}{2} + i\right)$$

Along $CO, x = 0$ then $dx = 0, z = iy, dz = idy$ and y varies from 1 to 0.

$$\therefore \int_{CO} (z+1)\ dz = \int_1^0 (iy+1)\, idy = \frac{1}{2} - i$$

Substituting the above four values in (1), we get

$$\int_C (z+1)\ dz = \frac{3}{2} + 2i - \frac{1}{2} - \left(\frac{3}{2} + i\right) + \frac{1}{2} - i = 0$$

EX.34. Verify Cauchy's theorem for the function $f(z) = z^2 + 3z - 2i$ if C is the circle $|z| = 1$.

Solution: Let $C: z = e^{i\theta}$ where $0 \leq \theta \leq 2\pi$.

$$\therefore \int_C f(z)\ dz = \int_0^{2\pi} \left(e^{2i\theta} + 3e^{i\theta} - 2i\right)\left(i.e^{i\theta}\right) d\theta$$

$$= i\int_0^{2\pi} \left(e^{3i\theta} + 3e^{2i\theta} - 2ie^{i\theta}\right) d\theta$$

$$= 0 \left(\int_0^{2\pi} e^{in\theta}\, d\theta = 0 \text{ if } n \neq 0\right)$$

Hence the theorem is verified.

2.7. Cauchy's Integral Formula:

If $f(z)$ is analytic within and on the closed curve C of a simply connected region R, and if a is any point in the interior of R, then

$$f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}dz,$$

Where the integration around C is in the positive sense.

Proof:

Given that $f(z)$ is analytic over a region R whose complete boundary is C.

Let a be any point inside R. Draw a circle C_0 with centre at a and radius r sufficiently small such that C_0 lies entirely in$R$. Since the function $f(z)$ is analytic everywhere within R, the function $\frac{f(z)}{z-a}$ is also analytic everywhere within R except at the one point $z = a$. In particular, $\frac{f(z)}{z-a}$ is analytic in the region R' between C and C_0. Hence the contour C may be deformed to the contour C_0. So applying Cauchy's extended theorem for the function $\frac{f(z)}{z-a}$, we have

$$\int_C \frac{f(z)}{z-a}dz = \int_{C_0} \frac{f(z)}{z-a}dz \qquad (1)$$

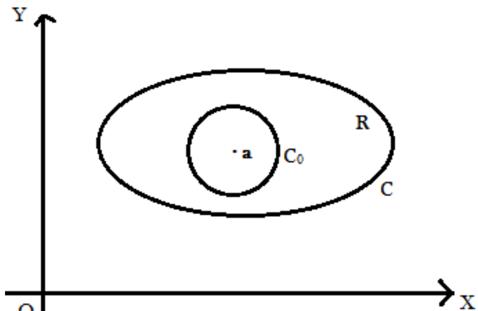


Figure 29

For a point z on C_0, we can put $z - a = r\,e^{i\theta}$.

Then $dz = r\,e^{i\theta}.id\theta$.

Therefore $$\int_{C_0} \frac{f(z)}{z-a}dz = \int_{C_0} \frac{f\left(a + r\,e^{i\theta}\right)}{r\,e^{i\theta}} r\,e^{i\theta}.id\theta$$

$$= i\int_{C_0} f\left(a + r\,e^{i\theta}\right)d\theta \qquad (2)$$

In the limits, as the circle C_0 reduces to the point a, as $r \to 0$.

Hence the integral in the right side of (2) approaches

$$i\int_{C_0} f(a)\,d\theta = if(a)\int_0^{2\pi} d\theta = if(a).2\pi$$

Substituting this value in (1), we get

$$\int_C \frac{f(z)}{z-a}dz = 2\pi i\, f(a)$$

$$\text{or}\qquad f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}dz \qquad (3)$$

The relation (3) is called Cauchy's Integral Formula. It expresses the value of an analytic function at an interior point of a region R in terms of its values on the boundary of the region.

2.8. Derivatives of an analytic function at interior points of a domain:

From Cauchy's integral formula, we can readily obtain an expression for the derivative of an analytic function at an interior point of R in terms of the boundary values of the function. By definition, taking $a$to be interior point and $f(z)$ as the analytic function, we have

$$f'(a) = \lim_{\Delta a\to 0}\frac{f(a+\Delta a) - f(a)}{\Delta a}$$

$$= \lim_{\Delta a\to 0}\frac{1}{\Delta a}\left[\frac{1}{2\pi i}\int_C \frac{f(z)}{z-(a+\Delta a)}dz - \frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}dz\right]$$

applying Cauchy's integral formula for both $f(a+\Delta a)$ and $f(a)$

$$= \lim_{\Delta a\to 0}\frac{1}{\Delta a}\left[\frac{1}{2\pi i}\int_C f(z)\left\{\frac{1}{z-(a+\Delta a)} - \frac{1}{z-a}\right\}dz\right]$$

$$= \lim_{\Delta a\to 0}\frac{1}{\Delta a}\left[\frac{1}{2\pi i}\int_C f(z)\left\{\frac{\Delta a}{(z-a-\Delta a)(z-a)}\right\}dz\right]$$

$$= \frac{1}{2\pi i}\times\lim_{\Delta a\to 0}\int_C \frac{f(z)}{(z-a-\Delta a)(z-a)}dz$$

$$= \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-a)^2}dz \qquad (1)$$

Proceeding similarly, we have

$$f''(a) == \frac{2!}{2\pi i}\int_C \frac{f(z)}{(z-a)^3}dz \qquad (2)$$

$$f'''(a) == \frac{3!}{2\pi i}\int_C \frac{f(z)}{(z-a)^4}dz \qquad (3)$$

and in general,

$$f^n(a) == \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-a)^{n+1}}dz \qquad (4)$$

This is called **Generalization of Cauchy's Integral Formula**.

We have thus established the important fact that analytic functions posses derivatives of all orders. Also we find that every derivative of an analytic function has a derivative and hence, in turn, is also analytic.

It can also be noted that the results for the derivatives $f'(a), f''(a), f'''(a)\ etc.$, obtained in $(1), (2), (3), \ldots..$ above can be obtained ordinarily by repeatedly differentiating within the integral sign Cauchy's integral formula with respect to the parameter a.

EXAMPLES

EX.35. Using Cauchy's integral formula, find the value of

$$\int_C \frac{z+4}{z^2+2z+5}dz$$

where C is the circle $|z+1-i| = 2$.

Solution: Given $|z+1-i| = 2, i.e., |z-(-1+i)| = 2$.

This is clearly a circle C with centre $-1+i$ and radius 2 units.

$$\frac{z+4}{z^2+2z+5} = \frac{z+4}{(z+1)^2+4} = \frac{z+4}{(z+1+2i)(z+1-2i)}$$

Consider the function

$$f(z) = \frac{z+4}{z+1+2i} \qquad (1)$$

This function is analytic at all points inside C. In fact, it is analytic everywhere except at $z = -1-2i$. but this point $(-1-2i)$ is outside C. The point $z = -1+2i$ is inside the circle C.

Hence by Cauchy's integral formula, we have

$$f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}dz$$

Taking $a = -1 + 2i$, we have

$$\int_C \frac{f(z)}{z-a}dz = 2\pi i. f(a)$$

$$i.e., \int_C \frac{z+4}{(z+1+2i)(z+1-2i)}dz = 2\pi i. f(a)$$

$$i.e., \qquad \int_C \frac{z+4}{z^2+2z+5}dz = 2\pi i. \frac{a+4}{a+1+2i} \qquad \text{[from (1)]}$$

$$= 2\pi i\left(\frac{-1+2i+4}{-1+2i+1+2i}\right)$$

$$= 2\pi i\left(\frac{3+2i}{4i}\right) = \frac{\pi}{2}(3+2i)$$

EX. 36. Prove that $\frac{1}{2\pi i}\int_C \frac{z^3-z}{(z-z_0)^3}dz = 3z_0$ if C is a closed curve described in the positive senseand z_0 is inside C. What will be its value when z_0 is outside C ?

Solution: By Cauchy's integration formula, we have

$$f^n(a) == \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-a)^{n+1}}dz$$

where a is a point inside C.

In this, take $f(z) = z^3 - z, a = z_0$ and $n = 2$. Then

$$f''(z_0) == \frac{2!}{2\pi i}\int_C \frac{z^3-z}{(z-z_0)^3}dz$$

$$\therefore \frac{1}{2\pi i}\int_C \frac{z^3-z}{(z-z_0)^3}dz = \frac{f''(z_0)}{2} \qquad (1)$$

Now differentiation gives, $f'(z) = 3z^2 - 1$ and $f''(z) = 6z$.

Therefore $f''(z_0) = 6z_0$ and (1) gives

$$\frac{1}{2\pi i}\int_C \frac{z^3-z}{(z-z_0)^3}dz = 3z_0$$

If z_0 is outside C, take $f(z) = \frac{z^3-z}{(z-z_0)^3}$. This function is analytic at all points inside C.

Therefore by Cauchy's theorem,

$$\int_C \frac{z^3 - z}{(z - z_0)^3} dz = 0$$

EX. 37. Evaluate $\oint_C \frac{z^2+4}{z-3}\, dz$ where C is $(a)|z| = 5 \quad (b)|z| = 2$ taken in anticlockwise (or positive sence).

Solution: $(a)|z| = 5$ is the circle with centre at (0, 0) and radius 5 units.

Given function is analytic everywhere except at $z = 3$ and lie inside C.

$$\oint_C \frac{z^2+4}{z-3}\, dz = \int_C \frac{f(z)}{z-a} dz$$

where $f(z) = z^2 + 4, a = 3$ and C is $|z| = 5$ taken in anticlockwise sese.

Using Cauchy's integral formula

$$\int_C \frac{f(z)}{z-a} dz = 2\pi i\, f(a) = 2\pi i[z^2 + 4]_{z=a=3}$$

$$= 2\pi i(9 + 4) = 26\pi i$$

$(b)|z| = 2$ is the circle with centre at (0, 0) and radius equal to 2.

The point $z = 3$ is outside this curve.

Therefore the function $\frac{z^2+4}{z-3}$ is analytic on and within $C{:}|z| = 2$.

Hence, by Cauchy's theorem

$$\oint_C \frac{z^2+4}{z-3}\, dz = 0.$$

EX.38. Let C be the circle $|z| = 3$ described in positive sense.

Let $g(a) = \int_C \frac{2z^2-z-2}{z-a} dz$ $(|a| \neq 3)$ Show that $g(2) = 8\pi i$. What is the value of $g(a)$ if $|a| > 3$.

Solution: $|z| = 3$ is the circle with centre at (0, 0) and radius equal to 3units.

Consider $g(a) = \int_C \frac{2z^2 - z - 2}{z - a} dz$

$\frac{2z^2 - z - 2}{z - a}$ is analytic everywhere except at $z = a$.

This point $z = a$ may be (i) within the circle or (ii) on the circle or (iii) outside the circle.

Since $|a| \neq 3$, $z = a$ is not on the circle.

Case I: If $z = a$is within the circle, $\frac{2z^2 - z - 2}{z - a}$ is analytic within C except at $z = a$.

Therefore take $f(z) = 2z^2 - z - 2$;

$$g(a) = \int_C \frac{2z^2 - z - 2}{z - a} dz$$

$$= \int_C \frac{f(z)}{z - a} dz = 2\pi i\, f(a) \text{ (by Cauchy's integral formula)}$$

$$= 2\pi i\, (2z^2 - z - 2) \text{at } z = a$$

$$= 2\pi i\, (2a^2 - a - 2)$$

$$\therefore g(2) = 2\pi i\, (8 - 2 - 2) = 8\pi i$$

Case II: If $|a| > 3$, $z = a$ is outside the circle $|z| = 3$.

Therefore $\frac{2z^2 - z - 2}{z - a}$ is analytic everywhere on and within C.

Hence $\int_C \frac{2z^2 - z - 2}{z - a} dz = 0$ by Cauchy's theorem.

EX. 39. Let C be a closed contour described in the positive sense.

Let $g(a) = \int_C \frac{z^3 + 2z}{(z - a)^3} dz$. Show that $g(a) = 6\pi i a$ if a is within C and $g(a) = 0$

when a is outside C.

Solution: **Case I**: Let $z = a$ be within C.

$$\text{Let } g(a) = \int_C \frac{z^3 + 2z}{(z - a)^3} dz$$

$$= \int_C \frac{f(z)}{(z - a)^3} dz$$

Using generalization to Cauchy's integral formula, we get

$$f^n(a) == \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-a)^{n+1}}dz$$

$\therefore f''(a) == \frac{2!}{2\pi i}\int_C \frac{f(z)}{(z-a)^{2+1}}dz$ where $f(z) = z^3 + 2z$

$f'(z) = 3z^2 + 2$ and $f''(z) = 6z$

$$\therefore \quad 6a = \frac{1}{\pi i}\int_C \frac{f(z)}{(z-a)^3}dz$$

$$i.e., \quad \int_C \frac{f(z)}{(z-a)^3}dz = 6\pi i a$$

$$i.e., \quad g(a) = 6\pi i a$$

Case II: Let $z = a$ be a point outside C.

Then the integrand in $\int_C \frac{z^3+2z}{(z-a)^3}dz$ is analytic on and within C everywhere.

Therefore by Cauchy's theorem, $\int_C \frac{z^3+2z}{(z-a)^3}dz = 0$

***EX*. 40.** Evaluate $\int_C \frac{dz}{2z-3}$, where C is the circle $|z| = 1$.

Solution: Cauchy's integral formula is

$$\int_C \frac{f(z)}{z-a}dz = 2\pi i\, f(a) \qquad (1)$$

Now $\int_C \frac{dz}{2z-3} = \frac{1}{2}\int_C \frac{dz}{z-\frac{3}{2}}$

Here $f(z) = 1, a = \frac{3}{2}$ which lies outside of the circle $|z| = 1$.

$$\therefore \int_C \frac{dz}{2z-3} = 0$$

EX. 41. Evaluate $\int_C \frac{dz}{2z+3}$, where C is the circle $|z| = 2$.

Solution: Cauchy's integral formula is

$$\int_C \frac{f(z)}{z-a} dz = 2\pi i\, f(a) \qquad (1)$$

Now $\int_C \frac{dz}{2z+3} = \frac{1}{2}\int_C \frac{dz}{z+\frac{3}{2}}$

Here $f(z) = 1, a = -\frac{3}{2}$ which lies inside of the circle $|z| = 2$.

$$\therefore \int_C \frac{dz}{2z-3} = \frac{1}{2}\int_C \frac{dz}{z+\frac{3}{2}} = \frac{1}{2}\, 2\pi i\, f\left(-\frac{1}{2}\right) \quad \text{[from (1)]}$$

$$= \pi i\,(1)\left(\because f(z) = 1 \Rightarrow f\left(-\frac{1}{2}\right) = 1\right)$$

$$= \pi i$$

***EX*. 42.** Evaluate $\int_C \frac{3z^2+7z+1}{z+1} dz$, where C is the circle $|z| = \frac{1}{2}$.

Solution: Given integrand is

$$\int_C \frac{3z^2+7z+1}{z+1} = \int_C \frac{3z^2+7z+1}{z-(-1)}$$

Here $f(z) = 3z^2 + 7z + 1, a = -1$ which lies outside of the circle $|z| = \frac{1}{2}$.

Therefore by Cauchy's theorem, we have

$$\therefore \int_C \frac{3z^2+7z+1}{z+1} = 0$$

***EX*. 43.** Evaluate $\int_C \frac{1}{ze^z} dz$, where C is the circle $|z| = 1$.

Solution: Given integrand is

$$\int_C \frac{1}{ze^z} dz = \int_C \frac{e^{-z}}{z-0} dz$$

Here $f(z) = e^{-z}, a = 0$ which lies inside of the circle $|z| = 1$.

Hence by Cauchy's integral theorem, we have

$$\int_C \frac{e^{-z}}{z-0}dz = 2\pi i\, f(0) = 2\pi i$$

EX. 44. Evaluate $\int_C \frac{e^{2z}}{(z-1)(z-2)}dz$, where C is the circle $|z| = 3$.

Solution: $f(z) = e^{2z}$ is analytic within the circle $C: |z| = 3$ and the two singular points $a = 1$ and $a = 2$ lie inside C.

$$\therefore \int_C \frac{e^{2z}}{(z-1)(z-2)}dz = \int_C e^{2z}\left(\frac{1}{z-2} - \frac{1}{z-1}\right)dz$$

$$= \int_C \frac{e^{2z}}{z-2}dz - \int_C \frac{e^{2z}}{z-1}dz$$

$$= 2\pi i\, f(2) - 2\pi i\, f(1) \text{ (by Cauchy's integral formula)}$$

$$= 2\pi i\, e^4 - 2\pi i\, e^2 = 2\pi i\,(e^4 - e^2)$$

EX. 45. Evaluate $\int_C \frac{\cos \pi z}{(z^2-1)}dz$, around a rectangle with vertices $2 \pm i, -2 \pm i$.

Solution: $f(z) = \cos \pi z$ is analytic in the region bounded by the given rectangle and the two singular points $a = 1$and $a = -1$ lie inside this rectangle.

$\therefore \int_C \frac{\cos \pi z}{(z^2-1)}dz = \frac{1}{2}\int_C \left(\frac{1}{z-1} - \frac{1}{z+1}\right)\cos \pi z\, dz$

$$= \frac{1}{2}\int_C \frac{\cos \pi z}{z-1}\, dz - \frac{1}{2}\int_C \frac{\cos \pi z}{z+1}\, dz$$

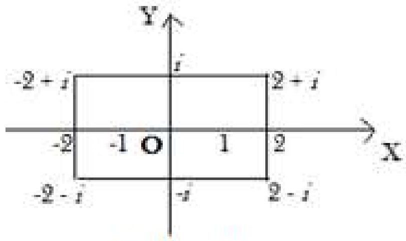


Figure 80

$= \frac{1}{2}\{2\pi i \cos \pi(1)\} - \frac{1}{2}\{2\pi i \cos \pi(-1)\} = 0$ [By Cauchy's i

EX. 46. Evaluate $\int_C \frac{\sin^2 z}{\left(z - \frac{\pi}{6}\right)^3}dz$, where C is the circle $|z| = 1$.

Solution: Let $f(z) = \sin^2 z$ is analytic inside the circle $C: |z| = 1$ and the point $a = \frac{\pi}{6}$ (0.5 approx.) lies within C.

Therefore by Cauchy's integral formula, we have

$$f''(a) = \frac{2!}{2\pi i}\int_C \frac{f(z)}{(z-a)^3}dz$$

$$\therefore \int_C \frac{sin^2 z}{\left(z-\frac{\pi}{6}\right)^3}dz = \pi i \left[\frac{d^2}{dz^2}(sin^2 z)\right]_{z=\frac{\pi}{6}}$$

$$= \pi i\,(2\,cos\,2z)_{z=\frac{\pi}{6}} = \pi i$$

EX. 47. Evaluate $\int_C \frac{e^{2z}}{(z+1)^4}dz$, where C is the circle $|z| = 2$.

Solution: Let $f(z) = e^{2z}$ is analytic within the circle $C: |z| = 2$. Also $z = -1$ lies inside C. Therefore by Cauchy's integral formula, we have

$$f'''(a) = \frac{3!}{2\pi i}\int_C \frac{f(z)}{(z-a)^4}dz$$

$$\therefore \int_C \frac{e^{2z}}{(z+1)^4}dz = \frac{2\pi i}{6}\left[\frac{d^3}{dz^3}(e^{2z})\right]_{z=-1}$$

$$= \frac{\pi i}{3}[8\,e^{2z}]_{z=-1} = \frac{8\pi i}{3}e^{-2}$$

EX. 48. Evaluate $\int_C \frac{e^z}{(z^2+\pi^2)^2}dz$, where C is the circle $|z| = 4$.

Solution: $\frac{e^z}{(z^2+\pi^2)^2} = \frac{e^z}{(z+\pi i)^2(z-\pi i)^2}$ is not analytic at $z = \pm\pi i$.

However both $z = \pm\pi i$ lie within the circle $|z| = 4$.

Now $\frac{e^z}{(z+\pi i)^2(z-\pi i)^2} = \frac{A}{z+\pi i} + \frac{B}{(z+\pi i)^2} + \frac{C}{z-\pi i} + \frac{D}{(z-\pi i)^2}$

where $A = \frac{7}{2\pi^3 i}$, $C = -\frac{7}{2\pi^3 i}$, $B = D = -\frac{1}{4\pi^2}$

$$\therefore \int_C \frac{e^z}{(z^2+\pi^2)^2}dz = \frac{7}{2\pi^3 i}\left\{\int_C \frac{e^z}{z+\pi i}dz - \int_C \frac{e^z}{z-\pi i}dz\right\}$$

$$-\frac{1}{4\pi^2}\left\{\int_C \frac{e^z}{(z+\pi i)^2}dz + \int_C \frac{e^z}{(z-\pi i)^2}dz\right\} =$$

$\frac{7}{2\pi^3 i}\{2\pi i\,f(-\pi i) - 2\pi i\,f(\pi i)\} - \frac{1}{4\pi^2}\{2\pi i\,f'(-\pi i) + 2\pi i\,f'(\pi i)\}$ here $f(z) = e^z$

$$= i/\pi$$

EX. 49. If $F(\zeta) = \int_C \frac{4z^2 + z + 5}{z - \zeta} dz$, where C is the ellipse $\left(\frac{x}{2}\right)^2 + \left(\frac{y}{2}\right)^2 = 1$,

find the value of (a) $F(3.5)$ (b) $F(i)$, $F'(-1)$ and $F''(-i)$

Solution: (a) $F(3.5) = \int_C \frac{4z^2 + z + 5}{z - 3.5} dz$

Since $\zeta = 3.5$ is the only singular point of $4z^2 + z + 5/(z - 3.5)$ and it lies outside the ellipse C, therefore, $4z^2 + z + 5/(z - 3.5)$ is analytic everywhere within C.

Hence by Cauchy's theorem, $\int_C \frac{4z^2 + z + 5}{z - 3.5} dz = 0, i.e., F(3.5) = 0$

(b) Since $f(z) = 4z^2 + z + 5$ is analytic within C and $\zeta = i, -1$ and $-\, i$ all lie within C, therefore,

by Cauchy's integral formula

$$f(\zeta) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - \zeta} dz$$

$$i.e., \int_C \frac{4z^2 + z + 5}{z - \zeta} dz = 2\pi i\, f(\zeta)$$

$$i.e., F(\zeta) = 2\pi i\, (4\zeta^2 + \zeta + 5)$$

$$\text{Then} F'(\zeta) = 2\pi i\, (8\zeta + 5) \text{and} F''(\zeta) = 16\pi i$$

$$\text{Thus } F(i) = 2\pi(i - 1)$$

$$F'(-1) = -14\, \pi i \text{ and} F''(-i) = 16\pi i$$

EX.50. Evaluate $\int_C \frac{z^3 - \sin 3z}{\left(z - \frac{\pi}{2}\right)^3} dz$, where C is the circle $|z| = 2$

using Cauchy's integral formula

Solution: $f(z) = z^3 - \sin 3z$ is analytic inside the circle $C: |z| = 2$ and the singular point $a = \frac{\pi}{2}$ lie inside C.

By Cauchy's integral formula, we have

$$f''(a) = \frac{2!}{2\pi i} \int_C \frac{f(z)}{(z - a)^3} dz$$

$$\therefore \int_C \frac{z^3 - \sin 3z}{\left(z - \frac{\pi}{2}\right)^3} dz = \pi i \, f''\left(\frac{\pi}{2}\right) = \pi i \, \frac{d^2}{dz^2}[z^3 - \sin 3z]_{z=\frac{\pi}{2}}$$

$$= \pi i \, \frac{d}{dz}[3z^2 - 3 \cos 3z]_{z=\frac{\pi}{2}}$$

$$= \pi i \, [6z + 9 \sin 3z]_{z=\frac{\pi}{2}}$$

$$= 3\pi i(\pi - 3)$$

EX. 51. Evaluate $\int_C \frac{ze^z}{(z+a)^3} dz$, where C is any simple closed curve enclosing the point $z = -a$.

Solution: Let $f(z) = ze^z$. Then $f(z)$ is analytic and the point $-a$ lies inside C.

By Cauchy's integral formula, we have

$$f''(a) = \frac{2!}{2\pi i} \int_C \frac{f(z)}{(z-a)^3} dz$$

$$\therefore \int_C \frac{f(z)}{(z+a)^3} dz = \pi i \, f''(-a) \qquad (1)$$

Now $f(z) = ze^z$

Then $f'(z) = ze^z + e^z$

$f''(z) = ze^z + 2e^z = (z+2)e^z$

$\therefore f''(-a) = (-a+2)e^{-a}$

Substituting the values of $f(z)$ and $f''(-a)$ in (1), we get

$$\int_C \frac{ze^z}{(z+a)^3} dz = (2-a) \, \pi i \, e^{-a}$$

EX. 52. Evaluate $\int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} dz$, where C is the circle $|z| = 3$ using Cauchy's integral formula.

Solution: Let $f(z) = \sin \pi z^2 + \cos \pi z^2$ is analytic within the circle $|z| = 3$ and the singular points $a = 1, 2$ lies inside C.

$$\therefore \int_C \frac{f(z)}{(z-1)(z-2)}dz = \int_C \left[\frac{1}{z-2} - \frac{1}{z-1}\right] f(z)dz$$

$$= \int_C \frac{f(z)}{z-2}dz - \int_C \frac{f(z)}{z-1}dz$$

$= 2\pi i\, f(2) - 2\pi i\, f(1)$(using Cauchy's integral formula)

$$= 2\pi i[(\sin 4\pi + \cos 4\pi) - (\sin \pi + \cos \pi)] \quad = 4\pi i$$

$$i.e., \int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)}dz = 4\pi i$$

EX. 53. Using Cauchy's integral formula, evaluate $\int_C \frac{z}{(z-1)(z-2)^2}dz$,

where C: $|z-2| = \frac{1}{2}$.

Solution: The integrand has two singular points at $z = 1$ and $z = 2$ of which only $z = 2$ lies inside C.

$f(z) = \frac{z}{z-1}$ is analytic on and within C.

Here $a = 2$ and $n = 1$.

Therefore by Cauchy' s integral formula

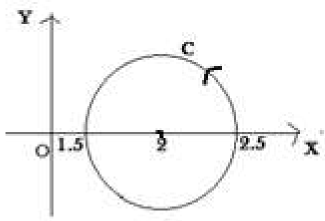


Figure 31

$$f'(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-a)^2}dz, \text{we get}$$

$$\int_C \frac{z}{(z-1)(z-2)^2}dz = 2\pi i \left[\frac{d}{dz}\left(\frac{z}{z-1}\right)\right]_{z=2}$$

$$= -2\pi i$$

***EX.* 54.** Use Cauchy's integral formula to evaluate $\int_C \frac{z^2-1}{z^2+1}dz$, where C: $|z-i| = 1$.

Solution: We have $\int_C \frac{z^2-1}{z^2+1}dz = \int_C \frac{z^2-1}{(z-i)(z+i)}dz$

The integrand has two singular points at $z = i$ and $z = -i$.

Among these only $z = i$ lies inside C.

Let $f(z) = \frac{z^2-1}{z+i}$ is analytic on and within C. Here $a = i$.

Therefore by Cauchy' s integral formula

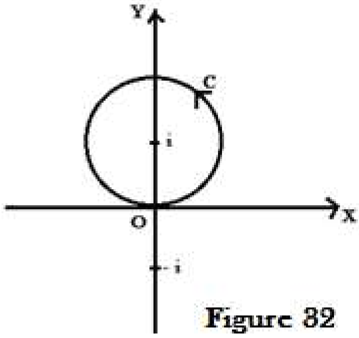


Figure 32

$$f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}dz, \text{we get}$$

$$\int_C \frac{z^2-1}{z^2+1}dz = 2\pi i\, f(a) = 2\pi i\, \left[\frac{z^2-1}{z+i}\right]_{z=i} = -2\pi.$$

EX. 55. Using Cauchy's integral formula, evaluate $\int_C \frac{z^2+1}{z(2z+1)}dz$, where C is $|z| = 1$.

Solution: The integrand has two singular points at $z = 0$ and $z = -\frac{1}{2}$.

Both lies inside the circle $|z| = 1$.

Now $\frac{1}{z(2z+1)} = \frac{1}{z} - \frac{2}{2z+1}$

$$\therefore \int_C \frac{z^2+1}{z(2z+1)}dz = \int_C \frac{z^2+1}{z}dz - 2\int_C \frac{z^2+1}{2z+1}dz$$

$= 2\pi i\, \left[f(0) - 2f\left(-\frac{1}{2}\right)\right]$ where $f(z) = z^2+1$, using Cauchy's integral formula

$= -3\pi i$

EX. 56. Using Cauchy's integral formula, evaluate $\int_C \frac{\cosh \pi z}{z(z^2+1)}dz$, where C is $|z| = 2$.

Solution: we have $\int_C \frac{\cosh \pi z}{z(z^2+1)}dz = \int_C \frac{\cosh \pi z}{z(z+i)(z-i)}dz$

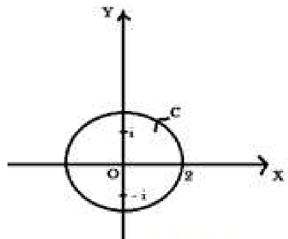


Figure 38

The integrand has three singular points at $z = 0, \pm i$.

By parital fractions, $\frac{\cosh \pi z}{z(z+i)(z-i)} = \frac{1}{z} - \frac{1}{2}\frac{1}{z-i} - \frac{1}{2}\frac{1}{z+i}$

Take $f(z) = \cosh \pi z$.

$$\therefore \int_C \frac{\cosh \pi z}{z(z^2+1)}dz = \int_C \frac{f(z)}{z}dz - \frac{1}{2}\int_C \frac{f(z)}{z-i}dz - \frac{1}{2}\int_C \frac{f(z)}{z+i}dz \text{ [from(1)and (2)]}$$

Here $a = 0, i, -i$.

Therefore by Cauchy' s integral formula $f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}dz$, we get

$$\int_C \frac{\cosh \pi z}{z(z^2+1)}dz = 2\pi i\, \left[f(0) - \frac{1}{2}f(i) - \frac{1}{2}f(-i)\right] = 4\pi i$$

EX. 57. If $F(\alpha) = \int_C \frac{5z^2 - 4z + 3}{z - \alpha} dz$, where C is the ellipse $\left(\frac{x}{3}\right)^2 + \left(\frac{y}{4}\right)^2 = 1$,

find the value of $(a) F(4.5)$ (b) $F(2), F'(i), F''(-2i)$.

Solution: We have $F(\alpha) = \int_C \frac{5z^2 - 4z + 3}{z - \alpha} dz$

(a) Taking $\alpha = 4.5$, we get $F(4.5) = \int_C \frac{5z^2 - 4z + 3}{z - 4.5} dz$

Now, the point $\alpha = 4.5$ lies outside the ellipse C.

Hence the function $\frac{5z^2 - 4z + 3}{z - 4.5}$ is analytic within and on C.

Therefore, by Cauchy's theorem, $\int_C \frac{5z^2 - 4z + 3}{z - 4.5} dz = 0$

$i.e.,\quad F(4.5) = 0$

(b) Let $f(z) = 5z^2 - 4z + 3$

Since $f(z)$ is analytic within C and $\alpha = 2, i, -2i$ all lie within C, therefore, by Cauchy's integral formula

$$f(\alpha) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - \alpha} dz$$

$$\therefore \qquad 2\pi i\, f(\alpha) = \int_C \frac{f(z)}{z - \alpha} dz = F(\alpha)$$

or $\quad F(\alpha) = 2\pi i\, f(\alpha) = 2\pi i\,(5\alpha^2 - 4\alpha + 3)$

$\therefore \quad F'(\alpha) = 2\pi i(10\alpha - 4)$ and $F''(\alpha) = 2\pi i(10) = 20\pi i$

Thus $F(2) = 2\pi i(20 - 8 + 3) = 30\pi i$

$F'(i) = 2\pi i(10i - 4) = -4\pi(5 + i2)$ and $F''(-2i) = 20\pi i$

EX. 58. Find $f(2)$ and $f(3)$ if $f(a) = \int_C \frac{2z^2 - z - 2}{z - a} dz$ where C is the circle $|z| = 2.5$

using Cauchy's integral formula.

Solution: Given $f(a) = \int_C \frac{2z^2 - z - 2}{z - a} dz$

(i) $a = 2$ lies inside the circle $C: |z| = 2.5$

Let $\phi(z) = 2z^2 - z - 2$

By Cauchy's integral formula, $\phi(a) = \frac{1}{2\pi i}\int_C \frac{\phi(z)}{z-a}dz$

$\Rightarrow\ 2\pi i\,\phi(a) = \int_C \frac{\phi(z)}{z-a}dz = f(a)$

$\Rightarrow\ f(a) = 2\pi i\,\phi(a) = 2\pi i\,(2a^2 - a - 2)$

$\therefore\ \ f(2) = 2\pi i\,(8 - 2 - 2) = 8\pi i$

(ii) Taking $a = 3$, we get, $f(3) = \int_C \frac{2z^2 - z - 2}{z-3}dz$

Now, the point $z = 3$ lies outside C. Hence the integrand is analytic within and on C.

$\therefore$ By Cauchy's theorem, $f(3) = \int_C \frac{2z^2 - z - 2}{z-3}dz = 0.$

EX. 59. Using Cauchy's integral formula, evaluate $\int_C \frac{z^4}{(z+1)(z-i)^2}dz$ where C is the ellipse $9x^2 + 4y^2 = 36$.

Solution: Given ellipse is $9x^2 + 4y^2 = 36$ *i.e.*, $\frac{x^2}{2^2} + \frac{y^2}{3^2} = 1$. Its centre is $(0, 0)$.

$f(z) = z^4$ is analytic within the ellipse $C: 9x^2 + 4y^2 = 36$ and the two singular points $a = -1, a = i$ lie inside C.

Consider $\frac{1}{(z+1)(z-i)^2}$

Let $\frac{1}{(z+1)(z-i)^2} = \frac{A}{z+1} + \frac{B}{z-i} + \frac{C}{(z-i)^2}$ (1)

Then $1 = A(z-i)^2 + B(z+1)(z-i) + C(z+1)$ (2)

$= (A+B)z^2 + (-2iA - iB + B + C)z + (-A - iB + C)$

Put $z = i$ in (2), we get

$1 = C(i+1) \qquad \Rightarrow C = \frac{1}{1+i}$

Put $z = -i$ in (2), we get

$1 = A(-i-1)^2 \quad \Rightarrow A = \frac{1}{(1+i)^2}$

Comparing constant term,

$$-A - iB + C = 1 \qquad \Rightarrow iB = -A + C - 1 = -\frac{1}{(1+i)^2} + \frac{1}{1+i} - 1$$

$$\Rightarrow iB = \frac{-i}{(1+i)^2} \text{ or } B = \frac{-1}{(1+i)^2}$$

Substituting the values of A, B and C in (1), we get

$$\frac{1}{(z+1)(z-i)^2} = \frac{1}{(1+i)^2}\frac{1}{z+1} + \frac{-1}{(1+i)^2}\frac{1}{z-i} + \frac{1}{1+i}\frac{1}{(z-i)^2}$$

Hence $\int_C \frac{z^4}{(z+1)(z-i)^2} dz$

$$= \frac{1}{(1+i)^2}\int_C \frac{z^4}{z+1} dz - \frac{1}{(1+i)^2}\int_C \frac{z^4}{z-i}\, dz + \frac{1}{1+i}\int_C \frac{z^4}{(z-i)^2} dz$$

$$= \frac{1}{(1+i)^2}.2\pi i.f(-1) - \frac{1}{(1+i)^2}.2\pi i.f(i) + \frac{1}{1+i}.2\pi i.f'(i)$$

(Using Cauchy's integral formula)

$$= 2\pi i\left[\frac{f(-1)}{(1+i)^2} - \frac{f(i)}{(1+i)^2} + \frac{f'(i)}{1+i}\right]$$

$$= 2\pi i\left[\frac{1}{(1+i)^2} - \frac{1}{(1+i)^2} + \frac{4i^3}{1+i}\right]$$

$$= 2\pi i\left(\frac{-4i}{1+i}\right) = \frac{8\pi}{1+i} = 4\pi(1-i)$$

INFINITE SERIES IN THE COMPLEX PLANE

2.9. Series of Complex Terms:

Most of the definitions and theorems relating to infinite series of real terms can be applied also to series whose terms are complex. Consider the infinite series

$$f_1(z) + f_2(z) + f_3(z) + \cdots + f_n(z) + \cdots \qquad (1)$$

whose terms are functions of the complex variable z.

Let $S_n(z)$ denote the sum of the first n terms of the above series. Then, if $S_n(z)$ tends to a finite limit $S(z)$ as n tends to infinity for all values of z in a region R, then the series is said to **converge** or to be **convergent** and to have the sum $S(z)$. R is called the region of convergence of the series. The difference $S(z) - S_n(z)$ is clearly the remainder after n terms of the series to be convergent, it is necessary that

$$\lim_{n\to\infty} S_n(z) - S(z) = 0$$

A series which is not convergent is said to **diverge** or to be **divergent**.

Now, the absolute values of the terms of (1) form another series

$$|f_1(z)| + |f_2(z)| + |f_3(z)| + \cdots + |f_n(z)| + \cdots \qquad (2)$$

If the series (2) is convergent, then the series (1) is said to be **absolutely convergent**. If (1) converges but (2) is not convergent, then it means that the series (1) is not absolutely convergent. It is only **conditionally convergent**.

2.10. Taylor's Series: Let $f(z)$ be analytic at all points within a circle C with centre at a and radius r. Then at each point z inside C,

$$f(z) = f(a) + f'(a).(z-a) + f''(a).\frac{(z-a)^2}{2!} + f'''(a).\frac{(z-a)^3}{3!} + \cdots + f^n(a).\frac{(z-a)^n}{n!} + \cdots$$

This is known as Taylor's series for the function $f(z)$.

Proof: For any point z in the interior of C, we can write Cauchy's integral formula as

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(z')}{z'-z}dz' (1)$$

$$\text{Now } \frac{1}{z'-z} = \frac{1}{(z'-a)-(z-a)} = \frac{1}{(z'-a)}\frac{1}{\left(1-\frac{z-a}{z'-a}\right)} \qquad (2)$$

Also we have the identity

$$1 + \alpha + \alpha^2 + \alpha^3 + \cdots + \alpha^{n-1} = \frac{1-\alpha^n}{1-\alpha},$$

where α is a complex number, not equal to 1.

$$i.e., 1 + \alpha + \alpha^2 + \alpha^3 + \cdots + \alpha^{n-1} + \frac{\alpha^n}{1-\alpha} = \frac{1}{1-\alpha} \qquad (3)$$

In equation (2), we substitute for

$$\frac{1}{1-\frac{z-a}{z'-a}}, \text{taking } \alpha = \frac{z-a}{z'-a} \text{ in equation (3).}$$

Then (2) gives

$$\frac{1}{z'-z} = \frac{1}{z'-a}\left[1 + \frac{z-a}{z'-a} + \left(\frac{z-a}{z'-a}\right)^2 + \cdots + \left(\frac{z-a}{z'-a}\right)^{\mathrm{n}-1} + \frac{1}{1-\frac{z-a}{z'-a}}\left(\frac{z-a}{z'-a}\right)^{\mathrm{n}}\right]$$

Therefore

$$\frac{f(z')}{z'-z} = \frac{f(z')}{z'-a} + \frac{(z-a)f(z')}{(z'-a)^2} + \frac{(z-a)^2 f(z')}{(z'-a)^3} + \cdots + \frac{(z-a)^{n-1} f(z')}{(z'-a)^n} + \frac{(z-a)^n f(z')}{(z'-z)(z'-a)^n}$$

We divide throughout by $2\pi i$ and integrate each term anticlockwise around C.

Therefore (1) becomes

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(z')}{z'-z}dz' = \frac{1}{2\pi i}\int_C \frac{f(z')}{z'-a}dz' + \frac{(z-a)}{2\pi i}\int_C \frac{f(z')}{(z'-a)^2}dz' + \cdots$$

$$+ \frac{(z-a)^{n-1}}{2\pi i}\int_C \frac{f(z')}{(z'-a)^n}dz' + \frac{(z-a)^n}{2\pi i}\int_C \frac{f(z')}{(z'-z)(z'-a)^n}dz' \qquad (4)$$

But we know from Cauchy's integral formula that

$$\frac{1}{2\pi i}\int_C \frac{f(z')}{z'-a}dz' = f(a)$$

$$\frac{1}{2\pi i}\int_C \frac{f(z')}{(z'-a)^2}dz' = f'(a)$$

$$\frac{1}{2\pi i}\int_C \frac{f(z')}{(z'-a)^3}dz' = \frac{f''(a)}{2!}$$

$$\text{and}\ \frac{1}{2\pi i}\int_C \frac{f(z')}{(z'-a)^n}dz' = \frac{f^{n-1}(a)}{(n-1)!}$$

Hence we can substitute these in the first n integrals on the right side of (4).

Therefore (4) becomes

$$f(z) = f(a) + f'(a).(z-a) + f''(a).\frac{(z-a)^2}{2!} + f'''(a).\frac{(z-a)^3}{3!} + \cdots$$

$$+ f^{n-1}(a).\frac{(z-a)^{n-1}}{(n-1)!} + \frac{(z-a)^n}{2\pi i}\int_C \frac{f(z')}{(z'-z)(z'-a)^n}dz' \qquad (5)$$

The difference between $f(z)$ and the sum of the first n terms is

$$= \frac{(z-a)^n}{2\pi i}\int_C \frac{f(z')}{(z'-z)(z'-a)^n}dz'$$

and this can be shown to approach zero as n tends to infinity. Hence as $n \to \infty$, the limit of the sum of the first n terms in the right side of (5) if $f(z)$.

Therefore $f(z)$ is represented by the infinite series

$$f(z) = f(a) + f'(a).(z-a) + f''(a).\frac{(z-a)^2}{2!} + f'''(a).\frac{(z-a)^3}{3!} + \cdots$$

$$= f(a) + \sum_{n=1}^{\infty} \frac{(z-a)^n}{n!} f^n(a) \qquad (6)$$

This is Taylor's series. It represents the function $f(z)$ at all points interior to any circle having its centre at a, and within which the function is analytic. The largest circle which can be drawn around $z = a$, such that $f(z)$ is analytic throughout its interior, is called the **circle of convergence** of the Taylor's series of $f(z)$. The radius of this circle is called the **radius of convergence** of the series.

Putting $a = 0$, (6) gives

$$f(z) == f(0) + \sum_{n=1}^{\infty} \frac{z^n}{n!} f^n(0)$$

$$i.e., f(z) == f(0) + z\, f'(0) + \frac{z^2}{2!} f''(0) + \cdots \qquad (7)$$

This is known as **Maclaurin's series**.

2.11. Standard Expansions: We have seen that when $f(z)$ is analytic at all points within the circle C, the Taylor's series of $f(z)$ is convergent within that circle. The maximum radius of C is the distance from the point a (the centre of C) to the singular point of $f(z)$ which is nearest to a, since the function is to be analytic at all points inside C.

The following are standard expansions which can be derived by using Maclaurin's theorem:

(1)$e^z = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \cdots \text{ when} |z| < \infty$

(2) $sin\, z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots \text{ when} |z| < \infty$

(3) $cos\, z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots \text{ when} |z| < \infty$

(4) $sinh\, z = z + \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots \text{ when} |z| < \infty$

(5) $cohs\, z = 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots \text{ when} |z| < \infty$

(6)$\frac{1}{1-z} = 1 + z + z^2 + \cdots \text{ when} |z| < 1$

2.12. Laurent's series: In most applications, we may require the expansion of a function around points, where, or in the neighbourhood of which, the functions are not analytic. Taylor's series is not obviously applicable in such cases and we use a new type of series known as **Laurent's series.** This series enables us to expand a function within an annular ring bounded by concentric circles, provided that the function which is being expanded is analytic everywhere between the circles. The function may have singular points outside the larger circle and also inside the smaller circle. In Laurent's expansion of such a function, there will be positive and negative powers of $z-a$. The expansion is given by the following theorem.

Theorem: If $f(z)$ is analytic throughout the ring shaped region R, bounded by two concentric circles C_1 and C_2 with centre a, then at any point z in the region R, $f(z)$ can be represented by a convergent series of positive and negative power of $z-a$.

$$i.e., f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \cdots + a_n(z-a)^n + \cdots$$
$$+\frac{b_1}{z-a} + \frac{b_2}{(z-a)^2} + \cdots + \frac{b_n}{(z-a)^n} + \cdots$$
$$= \sum_{n=0}^{\infty} a_n(z-a)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-a)^n}$$

$$\text{where} a_n = \frac{1}{2\pi i}\int_C \frac{f(z')}{(z'-a)^{n+1}}dz', n = 0,1,2,\ldots..$$

$$\text{and} b_n = \frac{1}{2\pi i}\int_C \frac{f(z')}{(z'-a)^{-n+1}}dz', n = 1,2,3,\ldots..$$

each integral being taken counterclockwise around any curve C, lying within the annulus and encircling its inner boundary.

Proof: Let $f(z)$ be analytic in the annular region R between two concentric circles C_1 and C_2. By making a crosscut joining any point of C_1 to any point of C_2 the annular region is converted into a region bounded by a single curve. Taking z to be an arbitrary point of the annulus, we have by Cauchy's integral formula,

$$f(z) = \frac{1}{2\pi i}\int_{C_1ABC_2BA} \frac{f(z')}{z'-z}dz'$$

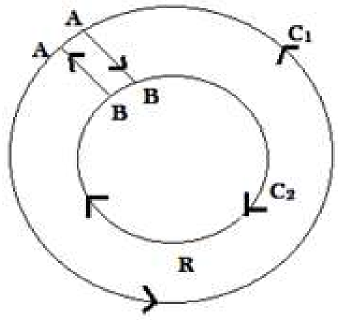


Figure 1

$$= \frac{1}{2\pi i}\left[\int_{C_1} + \int_{AB} + \int_{C_2} + \int_{BA}\right]\frac{f(z')}{z'-z}dz'$$

$$= \frac{1}{2\pi i}\left[\int_{C_1} + \int_{C_2}\right]\frac{f(z')}{z'-z}dz'$$

(since the integrals along AB and BA are cancel)

$$= \frac{1}{2\pi i}\int_{C_1}\frac{f(z')}{z'-z}dz' - \frac{1}{2\pi i}\int_{C_2}\frac{f(z')}{z'-z}dz'$$

where the integrations along C_1 and C_2 are both in the anticlockwise direction

$$= \frac{1}{2\pi i}\int_{C_1}\frac{f(z')}{(z'-a)-(z-a)}dz' + \frac{1}{2\pi i}\int_{C_2}\frac{f(z')}{(z-a)-(z'-a)}dz'$$

$$= \frac{1}{2\pi i}\int_{C_1}\frac{f(z')}{(z'-a)}\left[\frac{1}{1-\frac{z-a}{z'-a}}\right]dz' + \frac{1}{2\pi i}\int_{C_2}\frac{f(z')}{(z-a)}\left[\frac{1}{1-\frac{z'-a}{z-a}}\right]dz' \qquad (1)$$

In each of the integrals in the right side of (1), let us apply the identity

$$\frac{1}{1-\alpha} = 1 + \alpha + \alpha^2 + \alpha^3 + \cdots + \alpha^{n-1} + \frac{\alpha^n}{1-\alpha}$$

which we used in deriving Taylor's series.

Then (1) becomes

$$f(z) = \frac{1}{2\pi i}\int_{C_1}\frac{f(z')}{(z'-a)}\left[1 + \frac{z-a}{z'-a} + \left(\frac{z-a}{z'-a}\right)^2 + \cdots + \left(\frac{z-a}{z'-a}\right)^{n-1}\right.$$

$$\left. + \left(\frac{z-a}{z'-a}\right)^n \frac{1}{1-\frac{z-a}{z'-a}}\right]dz'$$

$$+\frac{1}{2\pi i}\int_{C_2}\frac{f(z')}{(z-a)}\left[1 + \frac{z'-a}{z-a} + \left(\frac{z'-a}{z-a}\right)^2 + \cdots + \left(\frac{z'-a}{z-a}\right)^{n-1} + \left(\frac{z'-a}{z-a}\right)^n \frac{1}{1-\frac{z'-a}{z-a}}\right]dz'$$

$$= \frac{1}{2\pi i}\int_{C_1}\frac{f(z')}{(z'-a)}dz' + \frac{z-a}{2\pi i}\int_{C_1}\frac{f(z')}{(z'-a)^2}dz' + \frac{(z-a)^2}{2\pi i}\int_{C_1}\frac{f(z')}{(z'-a)^3}dz' + \cdots$$

$$+ \frac{(z-a)^{n-1}}{2\pi i}\int_{C_1}\frac{f(z')}{(z'-a)^n}dz' + \frac{(z-a)^n}{2\pi i}\int_{C_1}\frac{f(z')}{(z'-a)^n(z'-z)}dz'$$

$$+\frac{1}{2\pi i(z-a)}\int_{C_2} f(z')\,dz' + \frac{1}{2\pi i(z-a)^2}\int_{C_2} f(z')\,(z'-a)dz'$$

$$+\frac{1}{2\pi i(z-a)^3}\int_{C_2} f(z')(z'-a)^2\,dz'$$

$$+\cdots+\frac{1}{2\pi i(z-a)^n}\int_{C_2} f(z')(z'-a)^{n-1}\,dz'+\cdots$$

$$+\frac{1}{2\pi i(z-a)^n}\int_{C_2}\frac{f(z')(z'-a)^n}{(z-z')}dz' \qquad (2)$$

$$= a_0 + a_1(z-a) + a_2(z-a)^2 + \cdots + a_n(z-a)^n + R_1$$

$$+\frac{b_1}{z-a}+\frac{b_2}{(z-a)^2}+\cdots+\frac{b_n}{(z-a)^n}+R_2 \qquad (3)$$

where $a_n = \frac{1}{2\pi i}\int_{C}\frac{f(z')}{(z'-a)^{n+1}}dz', n = 0, 1, 2,$ (4)

$$b_n = \frac{1}{2\pi i}\int_{C} f(z')(z'-a)^{n-1}\,dz', n = 1, 2, 3, \ldots..$$

$$i.e., b_n = \frac{1}{2\pi i}\int_{C}\frac{f(z')}{(z'-a)^{-n+1}}dz', n = 1, 2, 3, \ldots.. \qquad (5)$$

$$R_1 = \frac{(z-a)^n}{2\pi i}\int_{C_1}\frac{f(z')}{(z'-a)^n(z'-z)}dz'$$

and $R_2 = \frac{1}{2\pi i(z-a)^n}\int_{C_1}\frac{f(z')(z'-a)^n}{(z-z')}dz'$

It can be proved that

$$\lim_{n\to\infty} R_1 = 0 \text{ and } \lim_{n\to\infty} R_2 = 0$$

Hence $f(z)$ is represented by the infinite series

$$f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \cdots + a_n(z-a)^n + \cdots$$

$$+\frac{b_1}{z-a}+\frac{b_2}{(z-a)^2}+\cdots+\frac{b_n}{(z-a)^n}+\cdots \qquad (6)$$

where a_n and b_n are given by formulas (4) and (5).

Since $f(z)$ is analytic throughout the region between C_1 and C_2, the paths of integration C_1 and C_2 can be replaced by any other curve C within this region and encircling C_2.

Note 1: The coefficient

$$a_n = \frac{1}{2\pi i}\int_C \frac{f(z')}{(z'-a)^{n+1}}dz'$$

cannot be replaced by $\frac{f^n(a)}{n!}$ as we did in Taylor's series, since $f(z)$ is not analytic throughout the interior of C_1 and hence we cannot apply Cauchy's general integral formula.

Note 2: In most cases, the coefficients of Laurent expansion of a given function are not found by using the above theorem. They are got by using various algebraic manipulations depending on the nature of the function. In other words, the *Laurent expansion of a function over a given annular is unique*.

EX.1. Obtain the Taylor series expansion of $f(z) = \frac{1}{z}$ about the point $z = 1$.

Solution: At $z = 1$, $f(z)$ is analytic.

The point $z = 0$ is the only singular point and is at a distance of 1 unit from $z = 1$.

Hence, the Taylor's series expansion of $f(z) = \frac{1}{z}$ about $z = 1$.

Put $z - 1 = w$ then $z = w + 1$.

$$\therefore\ f(z) = \frac{1}{z} = \frac{1}{1+w} = (1+w)^{-1}$$

$$= 1 - w + w^2 - w^3 + \cdots \text{for} |w| < 1$$

$$= 1 - (z-1) + (z-1)^2 - (z-1)^3 + \cdots \text{for} |z-1| < 1$$

This is the required expansion.

EX.2. (i) Expand e^zas Taylor's series about $z = 1$.

(ii) Find the Taylor's series expansion of e^z about $z = 3$.

Solution: (i) We want the Taylor's series expansion of e^z around $z = 1$.

Put $z - 1 = w$ then $z = 1 + w$.

$$\therefore\ e^z = e^{1+w} = e.e^w$$

$$= e\left[1 + w + \frac{w^2}{2!} + \frac{w^3}{3!} + \cdots\right] \text{for all } w$$

$$= e\left[1 + (z-1) + \frac{(z-1)^2}{2!} + \frac{(z-1)^3}{3!} + \cdots\right]$$

This can also be written as $e^z = e + e\sum_{n=1}^{\infty}\frac{(z-1)^n}{n!}$ if $|z-1| < \infty$ $i.e.,$ for all z.

(ii) Proceeding as in (i), we obtain

$$e^z = e^{3+w} = e^3.e^w = e^3\left[1 + w + \frac{w^2}{2!} + \frac{w^3}{3!} + \cdots\right]$$

$$= e\left[1 + (z-3) + \frac{(z-3)^2}{2!} + \frac{(z-3)^3}{3!} + \cdots\right] \text{if} |z-1| < \infty \; i.e., \text{for all } z.$$

EX.3. Within what circle does the Maclaurin's series for the function $tanh\, z$ converge to the function.

Solution: $f(z) = tanh\, z = \frac{sinh\, z}{cosh\, z}$

The function is not analytic whenever $cosh\, z = 0$

$i.e., z = \pm\frac{\pi i}{2}, \pm\frac{3\pi i}{2}\; etc.$

We note that $f(z) = tanh\, z$ is analytic at $z = 0$ and the singular points $\pm\frac{\pi i}{2}$ are the nearest to $z = 0$ and are at a distance of $\frac{\pi}{2}$ from $z = 0$.

Hence, the Maclaurin's series expansion of $tanh\, z$ will be valid for the region $|z| < \frac{\pi}{2}$.

EX.4. Give two Laurent series expansion in powers of z, for the function $f(z) = \frac{1}{z^2(1-z)}$ and specify the regions in which those expansions are valid.

Solution: Case 1. $\frac{1}{z^2(1-z)} = \frac{1}{z^2}(1-z)^{-1}$

$$= \frac{1}{z^2}(1 + z + z^2 + z^3 + \cdots)$$

using the binomial theorem and taking $|z| < 1$

$$= \frac{1}{z^2} + \frac{1}{z} + 1 + z + z^2 + z^3 + \cdots$$

$$= z^{-2} + z^{-1} + z^0 + z^1 + z^2 + z^3 + \cdots$$

$$= \sum_{n=0}^{\infty} z^{n-2}$$

Clearly the series is valid in the region $0 < |z| < 1$.

Case 2. If $|z| > 1$, then $\left|\frac{1}{z}\right| < 1$. So we write

$$\frac{1}{z^2(1-z)} = -\frac{1}{z^2(z-1)} = -\frac{1}{z^3\left(1-\frac{1}{z}\right)}$$

$$= -\frac{1}{z^3}\left(1-\frac{1}{z}\right)^{-1}$$

$$= -\frac{1}{z^3}\left(1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\cdots\right)$$

$$= -\frac{1}{z^3}-\frac{1}{z^4}-\frac{1}{z^5}-\cdots$$

$$= -\sum_{n=0}^{\infty}\frac{1}{z^{n+3}}$$

This series is valid in the region $\left|\frac{1}{z}\right| < 1, i.e., |z| > 1$.

EX.5. Obtain the expansion of the function $\frac{z-1}{z^2}$ in (a) Taylor's series in powers of $z-1$ and give the region of validity (b) Laurent's series for the domain $|z-1| > 1$.

Solution: (a) Let $f(z) = \frac{z-1}{z^2}$

Then Taylor's series for $f(z)$ in powers of $z-1$ will be

$$f(z) = f(1) + \sum_{n=1}^{\infty}\frac{f^n(1)}{n!}(z-1)^n$$

Now $f(z) = \dfrac{z-1}{z^2} = \dfrac{1}{z} - \dfrac{1}{z^2}$ and $f(1) = 0$.

Differentiating n times,

$$f^n(z) = \frac{(-1)^n\, n!}{z^{n+1}} + \frac{(-1)^{n+1}(n+1)!}{z^{n+2}}$$

Therefore $f^n(1) = (-1)^n\, n! + (-1)^{n+1}(n+1)!$

$$= (-1)^n\, n!\,[1+(-1)(n+1)]$$

$$= (-1)^n\, n!\,(-n)$$

$$= (-1)^{n+1}\, n!\, n$$

Hence $f(z) = 0 + \displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\, n!\, n}{n!}(z-1)^n$

$$= \sum_{n=1}^{\infty}(-1)^{n+1}\; n\,(z-1)^n$$

$f(z)$ is not analytic at $z = 0$. Hence the above Taylor's series is convergent inside a circle with centre at $z = 1$ and radius <1. i.e., in the region $|z-1| < 1$.

(b) To get a Laurent's expansion in the region $|z-1| > 1$, we note that $\frac{1}{|z-1|} < 1$.

Hence expand $f(z)$ in powers of $\frac{1}{z-1}$.

$$f(z) = \frac{z-1}{z^2} = \frac{z-1}{(z-1+1)^2} = \frac{z-1}{\left[(z-1)\left(1+\frac{1}{z-1}\right)\right]^2}$$

$$= \frac{1}{z-1}\left[1+\frac{1}{z-1}\right]^{-2}$$

$$= \frac{1}{z-1}\left[1-\frac{2}{z-1}+\frac{3}{(z-1)^2}-\cdots+\frac{(-1)^{n-1}}{(z-1)^{n-1}}+\cdots\right]$$

$$= \frac{1}{z-1}-\frac{2}{(z-1)^2}+\frac{3}{(z-1)^3}-\cdots+\frac{(-1)^{n-1}n}{(z-1)^n}+\cdots$$

$= \sum_{n=1}^{\infty}(-1)^{n-1}\; n\,(z-1)^{-n}$ and this is valid in $|z-1| > 1$.

EX.6. Find the Laurent expansion of the function

$$f(z) = \frac{7z-2}{(z+1)\,z\,(z-2)}$$

In the annular $1 < |z+1| < 3$.

Solution: Put $z+1 = u$, then $z = u-1$.

Therefore

$$f(z) = \frac{7(u-1)-2}{u\,(u-1)(u-3)} = \frac{7u-9}{u\,(u-1)(u-3)}$$

$$= -\frac{3}{u}+\frac{1}{u-1}+\frac{2}{u-3}$$

$$= -\frac{3}{u}+\frac{1}{u\left(1-\frac{1}{u}\right)}-\frac{2}{3\left(1-\frac{u}{3}\right)}$$

$$= -\frac{3}{u}+\frac{1}{u}\left(1-\frac{1}{u}\right)^{-1}-\frac{2}{3}\left(1-\frac{u}{3}\right)^{-1}$$

$$= -\frac{3}{u}+\frac{1}{u}\left(1+\frac{1}{u}+\frac{1}{u^2}+\frac{1}{u^3}+\cdots\right)-\frac{2}{3}\left(1+\frac{u}{3}+\frac{u^2}{3^2}+\frac{u^3}{3^3}+\cdots\right)$$

$$= \left(-\frac{2}{u}+\frac{1}{u^2}+\frac{1}{u^3}+\cdots\right)-\frac{2}{3}\left(1+\frac{u}{3}+\frac{u^2}{3^2}+\frac{u^3}{3^3}+\cdots\right)$$

$$= \left(-\frac{2}{z+1}+\frac{1}{(z+1)^2}+\frac{1}{(z+1)^3}+\cdots\right)$$

$$-\frac{2}{3}\left(1+\frac{z+1}{3}+\frac{(z+1)^2}{3^2}+\frac{(z+1)^3}{3^3}+\cdots\right)$$

Clearly this series is valid in the region $\left|\frac{1}{u}\right| < 1$ and $\left|\frac{u}{3}\right| < 1$

i.e., $|u| > 1$ and $|u| < 3$, i.e., $1 < |u| < 3$

i.e., in the annulus $1 < |z+1| < 3$.

***EX.* 7.** Expand $f(z) = \dfrac{z-1}{z+1}$ in Taylor's series about the point $(i)\ z = 0\ \ (ii)\ z = 1$.

Solution: $(i)\ f(z) = \dfrac{z-1}{z+1} = \dfrac{z+1-2}{z+1} = 1 - \dfrac{2}{z+1} = 1 - 2.\dfrac{1}{1+z} = 1 - 2(1+z)^{-1}$

$$= 1 - 2(1 - z + z^2 - z^3 + \cdots) \text{ if } |z| < 1$$

$$= -1 + 2(z - z^2 + z^3 - \cdots) \text{ if } |z| < 1$$

$$= -1 + 2\sum_{n=1}^{\infty}(-1)^n z^n \text{ if } |z| < 1$$

(ii) To expand $f(z)$ about $z = 1$.

Put $z - 1 = w$, then $z = 1 + w$

Hence $f(z) = \dfrac{z-1}{z+1} = \dfrac{w}{1+w+1} = \dfrac{w}{2+w} = \dfrac{w}{2\left(1+\frac{w}{2}\right)} = \dfrac{w}{2}\left(1+\dfrac{w}{2}\right)^{-1}$

$$= \frac{w}{2}\left[1 - \frac{w}{2} + \left(\frac{w}{2}\right)^2 - \left(\frac{w}{2}\right)^3 + \cdots\right] \text{ if } \left|\frac{w}{2}\right| < 1$$

$$= \frac{w}{2} - \left(\frac{w}{2}\right)^2 + \left(\frac{w}{2}\right)^3 - \left(\frac{w}{2}\right)^4 + \cdots \text{ if } \left|\frac{w}{2}\right| < 1$$

$$= \sum_{n=1}^{\infty}(-1)^{n-1}\left(\frac{z-1}{2}\right)^n \text{ if } |z-1| < 2$$

EX.8. Find Taylor's expansion for the function $f(z) = \frac{1}{(1+z)^2}$ with centre at $-i$.

Solution: By Taylor's theorem

$$f(z) = f(a) + (z-a)f'(a) + \frac{(z-a)^2}{2!}f''(a) + \cdots + \frac{(z-a)^n}{n!}f^n(a) + \cdots$$

Put $a = -i$, then

$$\therefore\ f(z) = f(-i) + (z+i)f'(-i) + \frac{(z+i)^2}{2!}f''(-i) + \cdots + \frac{(z+i)^n}{n!}f^n(-i) + \cdots \quad (1)$$

Here $f(z) = \dfrac{1}{(1+z)^2}$

$$\therefore \quad f^n(z) = (-1)^n \frac{(n+1)!}{(1-i)^{n+2}}$$

$$\therefore \quad f(-i) = \frac{1}{(1+z)^2} = \frac{i}{2} \text{ and } f^n(-i) = (-1)^n \frac{(n+1)!}{(1-i)^{n+2}}$$

Substituting in (1), we get

$$\frac{1}{(1+z)^2} = \frac{i}{2} + \sum_{n=1}^{\infty} \frac{(z+i)^n}{n!} \frac{(-1)^n\,(n+1)!}{(1-i)^{n+2}} = \frac{i}{2} + \sum_{n=1}^{\infty} (-1)^n\,(n+1) \frac{(z+i)^n}{(1-i)^{n+2}}$$

$$= \frac{i}{2} + \sum_{n=1}^{\infty} (-1)^n\,(n+1) \frac{(z+i)^n}{(1-i)^n} \frac{1}{(1-i)^2}$$

$$= \frac{i}{2} + \sum_{n=1}^{\infty} (-1)^n\,(n+1) \frac{(z+i)^n}{(1-i)^n} \frac{1}{(-2i)}$$

$$= \frac{i}{2} + \frac{i}{2} \sum_{n=1}^{\infty} (-1)^n\,(n+1) \frac{(z+i)^n}{(1-i)^n}$$

EX.9. Obtain the Taylor expansion of $e^{(1+z)}$ in the powers of $(z-1)$.

Solution: By Taylor's theorem $f(z) = \sum_{n=0}^{\infty} \frac{(z-a)^n}{n!} f^n(a)$

Taking $a = 1$ and $f(z) = e^{(1+z)}$, we get

$$e^{(1+z)} = \sum_{n=0}^{\infty} \frac{(z-1)^n}{n!} e^{1+1} \left[\because\ f^n(z) = e^{(1+z)}\right] = e^2 \sum_{n=0}^{\infty} \frac{(z-1)^n}{n!}$$

which is the required Taylor's series.

EX.10. Obtain the Taylor's series to represent the function

$\frac{z^2-1}{(z+2)(z+3)}$, in the region $|z| < 2$.

Solution: Let $f(z) = \frac{z^2-1}{(z+2)(z+3)}$

$$= 1 + \frac{3}{z+2} - \frac{8}{z+3} \text{ (Resolving into partial fractions)}$$

$$= 1 + \frac{3}{2\left(1+\frac{z}{2}\right)} - \frac{8}{3\left(1+\frac{z}{3}\right)}$$

$$= 1 + \frac{3}{2}\left(1+\frac{z}{2}\right)^{-1} - \frac{8}{3}\left(1+\frac{z}{3}\right)^{-1}$$

Expanding by binomial series

$$f(z) = 1 + \frac{3}{2}\left(1 - \frac{z}{2} + \frac{z^2}{4} - \frac{z^3}{8} + \cdots\right) - \frac{8}{3}\left(1 - \frac{z}{3} + \frac{z^2}{9} - \frac{z^3}{27} + \cdots\right)$$

$$= 1 + \frac{3}{2}\sum_{n=0}^{\infty}\frac{(-1)^n}{2^n}.z^n - \frac{8}{3}\sum_{n=0}^{\infty}\frac{(-1)^n}{3^n}.z^n$$

$$= 1 + \frac{3}{2}\sum_{n=0}^{\infty}(-1)^n\left[\frac{3}{2^{n+1}} - \frac{8}{3^{n+1}}\right]z^n$$

which is the required Taylor's series.

EX.11. Find the Taylor's series expansion of $cosh\, z$ about $z = \pi i$.

Solution: Let $f(z) = cosh\, z$

Put $w = z - \pi i$. Then $z = w + \pi i$

$\therefore \quad f(z) = cosh\,(w + \pi i) = cosh\, w\, cosh\, \pi i - sinh\, w\, sinh\, \pi i$

But $cosh\, \pi i = cos\, \pi \left[\text{or } \frac{e^{\pi i} + e^{-\pi i}}{2} = \frac{2\,cos\,\pi}{2}\right] = -1$

and $sinh\, \pi i = i\, sin\, \pi \left[\text{or } \frac{e^{\pi i} - e^{-\pi i}}{2} = \frac{2\,sin\,\pi}{2}\right] = 0$

$\therefore \quad f(z) = -cosh\, w = -\left[1 + \frac{w^2}{2!} + \frac{w^4}{4!} + \cdots\right]$ for all w

$$= -\sum_{n=0}^{\infty}\frac{w^{2n}}{2n!} = -\sum_{n=0}^{\infty}\frac{(z-\pi i)^{2n}}{2n!}$$

which is the required Taylor's series.

Another Method:

$$f(z) = cosh\, z = \frac{e^z + e^{-z}}{2} = \frac{e^{z-\pi i+\pi i} + e^{-z+\pi i-\pi i}}{2}$$

$$= \frac{1}{2}e^{\pi i}.e^{z-\pi i} + \frac{1}{2}e^{-\pi i}.e^{-(z-\pi i)}$$

$$= -\frac{1}{2}\sum_{n=0}^{\infty}\frac{(z-\pi i)^n}{n!} - \frac{1}{2}\sum_{n=0}^{\infty}\frac{(-1)^n(z-\pi i)^n}{n!}$$

$$= -\sum_{n=0}^{\infty}\frac{(z-\pi i)^{2n}}{2n!}$$

EX.12. Expand $log\,(1-z)$ when $|z| < 1$ using Taylor series.

Solution: Let $f(z) = log\,(1-z) \qquad \therefore \quad f(0) = log\, 1 = 0$

Also $f'(z) = \dfrac{-1}{1-z}$ $\quad\therefore\quad f'(0) = -1$

$f''(z) = \dfrac{-1}{(1-z)^2}$ $\quad\therefore\quad f''(0) = -1$

$f'''(z) = \dfrac{-2}{(1-z)^3}$ $\quad\therefore\quad f'''(0) = -2$ and so on.

By Taylor's theorem about $z = 0$ is

$$f(z) = f(0) + zf'(0) + \frac{z^2}{2!}f''(0) + \frac{z^3}{3!}f'''(0) + \cdots$$

$$= 0 - z - \frac{z^2}{2!} - 2\frac{z^3}{3!} + \cdots$$

$$= -\left(z + \frac{z^2}{2} + \frac{z^3}{3} + \cdots\right)$$

which is the required Taylor's series.

***EX.* 13.** Find Taylor's expansion of $f(z) = \dfrac{2z^3+1}{z^2+z}$ about the point $(i)\ z = i\ \ (ii)\ z = 1$.

Solution: Given $f(z) = \dfrac{2z^3+1}{z^2+z} = 2z - 2 + \dfrac{2z+1}{z(z+1)}$ (Resolving into partial fractions)

$$= 2(z-1) + \frac{2}{z+1} + \frac{1}{z(z+1)}$$

$$= 2(z-1) + \frac{2}{z+1} + \frac{1}{z} - \frac{1}{z+1}$$

$$= 2(z-1) + \frac{1}{z} + \frac{1}{z+1} \qquad (1)$$

Differentiating (1) 'n' times,

$$f^n(z) = (-1)^n\, n!\left[\frac{1}{z^{n+1}} + \frac{1}{(z+1)^{n+1}}\right] \qquad (2)$$

$$f^n(i) = (-1)^n\, n!\left[\frac{1}{i^{n+1}} + \frac{1}{(i+1)^{n+1}}\right] \qquad (3)$$

By Taylor's theorem, $\quad f(z) = f(a) + \displaystyle\sum_{n=1}^{\infty}\frac{(z-a)^n}{n!}f^n(a) \qquad (4)$

(i) To find the Taylor's series about $z = i$

Putting $a = i$, we get

$$f(z) = f(i) + \sum_{n=1}^{\infty}\frac{(z-a)^n}{n!}f^n(i)$$

$$= \frac{i}{2} - \frac{3}{2} + \sum_{n=1}^{\infty} (-1)^n (z-i)^n \left[\frac{1}{i^{n+1}} + \frac{1}{(i+1)^{n+1}}\right], \text{using (3)}$$

$$= \frac{i}{2} - \frac{3}{2} + (z-i)\left(3 + \frac{i}{2}\right) + \cdots$$

(ii) To find the Taylor's series about $z = 1$

Singularities of $f(z)$ are given by $z = 0$ and $z = -1$. Draw a circle with centre at $z = 1$ and radius 1. Then within the circle $|z-1| = 1$, the function $f(z)$ is analytic. Thus $f(z)$ can be expanded in a Taylor's series within the circle $|z-1| = 1$, which is the circle of convergence.

From (2), we have

$$f^n(z) = (-1)^n\, n!\left[\frac{1}{z^{n+1}} + \frac{1}{(z+1)^{n+1}}\right]$$

$$\therefore\ f^n(1) = (-1)^n\, n!\left[\frac{1}{1^{n+1}} + \frac{1}{2^{n+1}}\right] = (-1)^n\, n!\left(1 + \frac{1}{2^{n+1}}\right)$$

By Taylor's theorem,

$$f(z) = f(1) + \sum_{n=1}^{\infty} \frac{(z-1)^n}{n!} f^n(1), \qquad \text{using (4)}$$

$$= \frac{3}{2} + \sum_{n=1}^{\infty} \frac{(z-1)^n}{n!} (-1)^n\, n!\left(1 + \frac{1}{2^{n+1}}\right)$$

$$= \frac{3}{2} + \sum_{n=1}^{\infty} (-1)^n \left(1 + \frac{1}{2^{n+1}}\right)(z-1)^n$$

EX.14. Expand $f(z) = sin\, z$ in Taylor's series about $z = \frac{\pi}{4}$.

Solution: By Taylor′s theorem,

$$f(z) = f(a) + (z-a)f'(a) + \frac{(z-a)^2}{2!} f''(a) + \cdots + \frac{(z-a)^n}{n!} f^n(a) + \cdots \qquad (1)$$

Put $a = \frac{\pi}{4}$ in (1), then

$$f(z) = f\left(\frac{\pi}{4}\right) + \left(z - \frac{\pi}{4}\right) f'\left(\frac{\pi}{4}\right) + \frac{\left(z-\frac{\pi}{4}\right)^2}{2!} f''\left(\frac{\pi}{4}\right) + \cdots + \frac{\left(z-\frac{\pi}{4}\right)^n}{n!} f^n\left(\frac{\pi}{4}\right) + \cdots \qquad (2)$$

Now $\quad f(z) = sin\, z \qquad f\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$

$f'(z) = cos\, z \qquad f'\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$

$$f''(z) = -sin\, z \qquad f''\left(\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}$$

$$f'''(z) = -sin\, z \qquad f'''\left(\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}} \text{ and so on.}$$

Substituting in (2), we get

$$f(z) = \frac{1}{\sqrt{2}} + \left(z - \frac{\pi}{4}\right)\frac{1}{\sqrt{2}} + \frac{\left(z-\frac{\pi}{4}\right)^2}{2!}\left(-\frac{1}{\sqrt{2}}\right) + \frac{\left(z-\frac{\pi}{4}\right)^3}{3!}\left(-\frac{1}{\sqrt{2}}\right) + \cdots$$

$$i.e., \qquad sin\, z = \frac{1}{\sqrt{2}}\left[1 + \left(z - \frac{\pi}{4}\right) - \frac{1}{2!}\left(z - \frac{\pi}{4}\right)^2 - \frac{1}{3!}\left(z - \frac{\pi}{4}\right)^3 + \cdots\right]$$

EX.15. Obtain the Taylor's series expansion of $f(z) = \frac{e^z}{z(z+1)}$ about $z = 2$.

Solution: Given $f(z) = \dfrac{e^z}{z(z+1)} = \dfrac{e^{z-2}.e^2}{(z-2+2)(z-2+3)}$

$$= e^2 \frac{e^{z-2}}{2\left[1 + \frac{z-2}{2}\right]} \cdot \frac{1}{3\left[1 + \frac{z-2}{3}\right]}$$

$$= \frac{e^2}{6}.e^{z-2}.\left(1 + \frac{z-2}{2}\right)^{-1}\left(1 + \frac{z-2}{3}\right)^{-1} \text{ valid when } \left|\frac{z-2}{2}\right| < 1 \text{ and } \left|\frac{z-2}{3}\right| < 1$$

$$= \frac{e^2}{6}\left[\sum_{n=0}^{\infty}\frac{(z-2)^n}{n!}\right]\left[\sum_{n=0}^{\infty}(-1)^n\left(\frac{z-2}{2}\right)^n\right]\left[\sum_{n=0}^{\infty}(-1)^n\left(\frac{z-2}{3}\right)^n\right]$$

$$= \frac{e^2}{6}\left[1 - \frac{z-2}{1!} + \frac{(z-2)^2}{2!} + \frac{(z-2)^3}{3!} + \cdots\right] \times$$

$$\left[1 - \frac{z-2}{2} + \frac{(z-2)^2}{4} - \frac{(z-2)^3}{8} + \cdots\right] \times$$

$$\left[1 - \frac{z-2}{3} + \frac{(z-2)^2}{9} - \frac{(z-2)^3}{27} + \cdots\right]$$

$$= \frac{e^2}{6}\left[1 - \frac{z-2}{2} + \frac{(z-2)^2}{4} + \frac{z-2}{1} - \frac{(z-2)^2}{2} + \frac{(z-2)^2}{2} + \cdots\right] \times$$

$$\left[1 - \frac{z-2}{3} + \frac{(z-2)^2}{9} + \cdots\right]$$

$$= \frac{e^2}{6}\left[1 + \frac{z-2}{2} + \frac{(z-2)^2}{4} + \cdots\right] \times \left[1 - \frac{z-2}{3} + \frac{(z-2)^2}{9} + \cdots\right]$$

$$= \frac{e^2}{6}\left[1 - \frac{z-2}{3} + \frac{(z-2)^2}{9} + \frac{z-2}{2} - \frac{(z-2)^2}{6} + \frac{(z-2)^2}{4} - \frac{(z-2)^3}{12} + \cdots\right]$$

$$= \frac{e^2}{6}\left[1 + \frac{z-2}{6} + \frac{7}{36}(z-2)^2 + \cdots\right]$$

EX. 16. Let $f(z) = \dfrac{1}{(1-z)(z-2)}$,

find Laurent′s series expansion in the annulus region $1 < |z| < 2$.

Also find the Laurent's series expansion in $|z| > 2$.

Solution: Let $f(z) = \dfrac{1}{(1-z)(z-2)} = \dfrac{-1}{(z-1)(z-2)}$

$$= \frac{(z-2)-(z-1)}{(z-1)(z-2)} = \frac{1}{z-1} - \frac{1}{z-2}$$

Laurent′s series expansion in the annulus region $1 < |z| < 2$.

$|z| = 2, |z| = 1$ are two concentric circles with centre at O and radii equal to 1 and 2 respectively. In $1 < |z| < 2$, $f(z)$ is analytic

$$f(z) = \frac{1}{z-1} - \frac{1}{z-2} = \frac{1}{z\left(1-\frac{1}{z}\right)} + \frac{1}{2\left(1-\frac{z}{2}\right)}$$

$$= \frac{1}{z}\left[1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \cdots\right] + \frac{1}{2}\left[1 + \frac{z}{2} + \left(\frac{z}{2}\right)^2 + \left(\frac{z}{2}\right)^3 + \cdots\right]$$

Here the first expansion is valid if $\left|\frac{1}{z}\right| < 1, i.e.,$ if $|z| > 1$ and the second expansion is valid if $\left|\frac{z}{2}\right| < 1, i.e.,$ if $|z| < 2$.

Hence, $f(z) = \left(\dfrac{1}{z} + \dfrac{1}{z^2} + \dfrac{1}{z^3} + \dfrac{1}{z^4} + \cdots\right) + \left(\dfrac{1}{2} + \dfrac{z}{2^2} + \dfrac{z}{2^3} + \dfrac{z}{2^4} + \cdots\right)$

$$= \sum_{n=-\infty}^{\infty} a_n z^n, where\ a_n = \begin{cases} \dfrac{1}{2^{n+1}}, & if\ n = 0,1,2,\ldots \\ 1, & if\ n = -1,-2,-3,\ldots \end{cases}$$

The expansion is valid if both $|z| > 1$ and $|z| < 2$ are true, i.e., if $1 < |z| < 2$.

Laurent′s series expansion in the annulus region $|z| > 2$.

$$f(z) = \frac{1}{z-1} - \frac{1}{z-2} = \frac{1}{z\left(1-\frac{1}{z}\right)} + \frac{1}{z\left(1-\frac{2}{z}\right)}$$

$$= \frac{1}{z}\left[1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \cdots\right] - \frac{1}{z}\left[1 + \frac{2}{z} + \left(\frac{2}{z}\right)^2 + \left(\frac{2}{z}\right)^3 + \cdots\right]\ if\ \left|\frac{1}{z}\right| < 1\ and\ \left|\frac{2}{z}\right| < 1$$

$$= \sum_{n=1}^{\infty} \frac{(1-2^n)}{z^{n+1}}\ i.e., if\ |z| > 1 \text{ and } |z| > 2$$

Hence, $f(z) = \sum_{n=1}^{\infty} \frac{(1-2^n)}{z^{n+1}}\ if\ |z| > 2.$

EX.17. Expand $f(z) = \frac{1}{(z-1)(z-2)}$, in the region $(i)\ 0 < |z-1| < 1\ (ii)\ 1 < |z| < 2.$

Solution: Given $f(z) = \frac{1}{(z-1)(z-2)} = \frac{1}{z-2} - \frac{1}{z-1}$

$z = 1, z = 2$ are the singular points of $f(z)$.

(i) The function $f(z)$ is analytic in the ring shaped region $0 < r_2 < |z-1| < r_1 < 1$.

Put $z - 1 = w \quad \therefore z = w + 1 \quad \Rightarrow z - 2 = w + 1 - 2 = w - 1$

$$\therefore \qquad f(z) = \frac{1}{w-1} - \frac{1}{w} = -\frac{1}{w} - \frac{1}{w-1}$$

$$= -\frac{1}{w} - [1 + w + w^2 + \cdots]\ if\ w < 1 \text{ and } w \neq 0$$

$$= -\frac{1}{z-1} - \sum_{n=0}^{\infty} (z-1)^n \text{ if } 0 < |z-1| < 1$$

$$= -\sum_{n=-1}^{\infty} (z-1)^n \text{ if } 0 < |z-1| < 1.$$

(ii) Given $1 < |z| < 2$ $i.e.$, $1 < |z|$ and $|z| < 2$ or $\left|\frac{1}{z}\right| < 1$ and $\left|\frac{z}{2}\right| < 1$

$$\therefore \qquad f(z) = \frac{1}{z-2} - \frac{1}{z-1} = \frac{1}{-2\left(1-\frac{z}{2}\right)} - \frac{1}{z\left(1-\frac{1}{z}\right)}$$

$$= -\frac{1}{2}\left(1-\frac{z}{2}\right)^{-1} - \frac{1}{z}\left(1-\frac{1}{z}\right)^{-1}$$

$$= -\frac{1}{2}\left[1 + \frac{z}{2} + \left(\frac{z}{2}\right)^2 + \left(\frac{z}{2}\right)^3 + \cdots\right] - \frac{1}{z}\left[1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \cdots\right]$$

$$= -\frac{1}{2}\left[1 + \frac{z}{2} + \frac{z^2}{4} + \frac{z^3}{8} + \cdots\right] - \left[\frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \cdots\right]$$

EX.18. Expand $f(z) = \frac{e^{2z}}{(z-1)^3}$ about $z = 1$

as a Laurent's series. Also find the region of convergence.

Solution: $f(z) = \frac{e^{2z}}{(z-1)^3}$

We want the Laurent's series expansion of $f(z)$ around $z = 1$.

Put $z - 1 = w \quad \therefore z = w + 1$

$$\therefore \qquad f(z) = \frac{e^{2z}}{(z-1)^3} = \frac{e^{2(1+w)}}{w^3} = e^2.\frac{e^{2w}}{w^3}$$

$$= e^2.\frac{1}{w^3}\left[1 + (2w) + \frac{(2w)^2}{2!} + \frac{(2w)^3}{3!} + \cdots\right]$$

$$= e^2 \sum_{n=0}^{\infty} \frac{2^n}{n!}(w^{n-3}) \text{ if } w \neq 0$$

$$= e^2 \sum_{n=0}^{\infty} \frac{2^n}{n!}(z-1)^{n-3} \text{ if } z-1 > 0$$

EX.19. Find the Laurent′s series expansion of the function $f(z) = \frac{z^2-6z-1}{(z-1)(z-3)(z+2)}$ in the region $3 < |z+2| < 5$.

Solution: By partial fractions

$$\frac{z^2-6z-1}{(z-1)(z-3)(z+2)} = \frac{1}{z-1} - \frac{1}{z-3} + \frac{1}{z+2}$$

$$= \frac{1}{z+2-3} - \frac{1}{z+2-5} + \frac{1}{z+2}$$

$$= \frac{1}{(z+2)\left(1-\frac{3}{z+2}\right)} + \frac{1}{5\left(1-\frac{z+2}{5}\right)} + \frac{1}{z+2}$$

$$= \frac{1}{(z+2)}\left(1-\frac{3}{z+2}\right)^{-1} + \frac{1}{5}\left(1-\frac{z+2}{5}\right)^{-1} + \frac{1}{z+2}$$

$$= \frac{1}{(z+2)}\sum_{n=0}^{\infty}\left(\frac{3}{z+2}\right)^n + \frac{1}{5}\sum_{n=0}^{\infty}\left(\frac{z+2}{5}\right)^n + \frac{1}{z+2}$$

valid for $\left|\frac{3}{z+2}\right| < 1$ and $\left|\frac{z+2}{5}\right| < 1 \;\Rightarrow\; 3 < |z+2| < 5.$

EX.20. Find the Laurent′s series of $f(z) = \frac{7z-2}{(z+1)z(z-2)}$ in the annulus $1 < |z+1| < 3$.

(OR)

Expand $f(z) = \dfrac{7z-2}{(z+1)z(z-2)}$ about the point $z = -1$ in the region $1 < |z+1| < 3$ as Laurent's series.

Solution: Let $f(z) = \dfrac{7z-2}{(z+1)z(z-2)}$

Put $z+1 = w$ Then $z = w-1$

$$\therefore \qquad f(z) = \frac{7(w-1)-2}{w(w-1)(w-1-2)} = \frac{7w-9}{w(w-1)(w-3)}$$

$$= -\frac{3}{w} + \frac{1}{w-1} + \frac{2}{w-3}, \text{ by partial fractions}$$

$$= -\frac{3}{w} + \frac{1}{w\left(1-\frac{1}{w}\right)} - \frac{2}{3\left(1-\frac{w}{3}\right)}$$

$$= -\frac{3}{w} + \frac{1}{w}\left(1-\frac{1}{w}\right)^{-1} - \frac{2}{3}\left(1-\frac{w}{3}\right)^{-1}$$

$$= -\frac{3}{w} + \frac{1}{w}\left(1+\frac{1}{w}+\frac{1}{w^2}+\cdots\right) - \frac{2}{3}\left(1+\frac{w}{3}+\frac{w^2}{3^2}+\cdots\right)$$

$$= \left(-\frac{2}{w}+\frac{1}{w^2}+\frac{1}{w^3}+\cdots\right) - \frac{2}{3}\left(1+\frac{w}{3}+\frac{w^2}{3^2}+\cdots\right)$$

$$= -\frac{2}{z+1} + \frac{1}{(z+1)^2} + \frac{1}{(z+1)^3} + \cdots - \frac{2}{3}\left[1+\frac{z+1}{3}+\frac{(z+1)^2}{3^2}+\cdots\right]$$

The above series valid for $\left|\frac{1}{z+1}\right| < 1$ and $\left|\frac{z+1}{3}\right| < 1 \;\Rightarrow\; 1 < |z+1| < 3.$

THE CALCULUS OF RESIDUES

2.13. **Singular Points of an Analytic Function**:

All the points of the z-plane at which an analytic function does not have a unique derivative are said to be **singular points**. If $z = a$ is a singular point of the function $f(z)$ such that there exists a circle with centre at a in which there are no other singular points of $f(z)$, then $z = a$ is called an isolated singular point of $f(z)$. This means that there is some neighbourhood of the singular point a of the function $f(z)$ throughout which it is analytic, except at the point itself. For instance, the function $\frac{1}{z}$ is analytic everywhere in the complex plane except at $z = 0$; hence the origin is an isolated singular point of the function. The function $\frac{z+2}{z(z^2-1)}$ has three isolated singular points, namely $z = 0$ and $z = \pm 1$.

2.14. Types of singularities:

Let $f(z)$ be analytic within a domain D, except at the point $z = a$, which is an isolated singularity of $f(z)$. We can draw two concentric circles of centre a, both lying within D. The radius r_2 of the smaller circle may be as small as we please and the radius r_1 of the larger circle may be of any length, subject to the condition that the circle lies within D. In the annulus between these two circles, we can expand $f(z)$ in a Laurent's series in powers of $z - a$; this expansion will contain both positive and negative powers of $z - a$. Let this expansion be

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} b_n (z-a)^{-n}$$

$$= a_0 + a_1(z-a) + a_2(z-a)^2 + \cdots + b_1(z-a)^{-1} + b_2(z-a)^{-2} + \cdots \quad (1)$$

The part

$$\sum_{n=0}^{\infty} a_n (z-a)^n$$

is called the analytic part while the part

$$\sum_{n=1}^{\infty} b_n (z-a)^{-n}$$

containing negative powers of $z - a$ is called the principal part of $f(z)$at the singular point $z = a$.

We consider the following cases:

Case I: Let all the coefficients b_n in the expansion (1) be zero i.e., there is no principal part in (1). The remaining terms are the Taylor series expansion and the function $f(z)$can be made analytic by suitably defining its value at a. We then call $z = a$ a ***removable singularity*** of $f(z)$.

Case II: Let the expansion (1) contain an infinite number of negative powers of $z - a$. In this case, the point $z = a$ said to be an ***essential singularity*** of $f(z)$.

Case III: If the principal part of the expansion (1) contains only the single term $\frac{b_1}{z-a}$, then the singularity at a is known as a ***simple pole*** or pole of order one. If the principal part contains a finite number of negative powers of $z - a$, and if b_m is the last non-zero coefficient in the principal part, then a is said to be a ***pole*** of order m. In such case, m is clearly the largest of the negative exponents. Poles of orders 1, 2, 3, …….. are usually called simple, double, triple, ……poles.

Note: Let $f(z)$have a pole of order m at $z = a$. Then Laurent series takes the form

$$f(z) = \sum_{n=0}^{\infty} a_n\,(z-a)^n + \frac{b_1}{z-a} + \frac{b_2}{(z-a)^2} + \cdots + \frac{b_m}{(z-a)^m}$$

$$= \frac{1}{(z-a)^m}\left[\sum_{n=0}^{\infty} a_n\,(z-a)^{m+n} + b_m + b_{m-1}(z-a) + \cdots + b_1(z-a)^{m-1}\right]$$

$$= \frac{1}{(z-a)^m}\emptyset(z)$$

where $\emptyset(z)$represents the series inside the bracket.

Now $\emptyset(a) = b_m$ and this is not equal to zero as $f(z)$ has a pole of order m.

Hence if a function

$$f(z) = \frac{\emptyset(z)}{(z-a)^m}$$

where $\emptyset(z)$ is analytic everywhere in a region including $z = a$ such that $\emptyset(a) \neq 0$, and if m is a positive integer, we can conclude that $f(z)$ has an isolated singularity at $z = a$ which is a pole of order m.

Example 1. Find the nature and location of singularities of the following functions:

$(i)\ \dfrac{z - sin\, z}{z^2} \qquad (ii)\ (z+1)\ sin\ \dfrac{1}{z-2} \qquad (iii)\ \dfrac{1}{cos\, z - sin\, z}$

Solution: (i) Here $z = 0$ is a singularity.

Also $\dfrac{z - sin\, z}{z^2} = \dfrac{1}{z^2}\left\{z - \left(z - \dfrac{z^3}{3!} + \dfrac{z^5}{5!} - \dfrac{z^7}{7!} + \cdots\right)\right\} = \dfrac{z}{3!} - \dfrac{z^3}{5!} + \dfrac{z^5}{7!} - \cdots$

Since there are no negative powers of z in the expansion, $z = 0$ is a removable singularity.

(ii) $\quad (z+1)\ sin\ \dfrac{1}{z-2} = (t+2+1)\ sin\ \dfrac{1}{t}$ where $t = z - 2$

$$= (t+3)\left\{\frac{1}{t} - \frac{1}{3!\,t^3} + \frac{1}{5!\,t^5} - \cdots\right\}$$

$$= \left(1 - \frac{1}{3!\,t^2} + \frac{1}{5!\,t^4} - \cdots\right) + \left(\frac{3}{t} - \frac{1}{2t^3} + \frac{3}{5!\,t^5} - \cdots\right)$$

$$= 1 + \frac{3}{t} - \frac{1}{6t^2} - \frac{1}{2t^3} + \frac{1}{120t^4} - ....$$

$$= 1 + \frac{3}{z-2} - \frac{1}{6(z-2)^2} - \frac{1}{2(z-2)^3 + ...}$$

Since there are infinite number of terms in the negative powers of (z-2), z=2 is an essential singularity.

(iii) Poles of $f(z) = \dfrac{1}{\cos z - \sin z}$ are given by equating the denominator to zero,

i.e., by *cos z–sin z=0* or *tan z = 1* or $z = \pi/4$. Clearly $z = \pi/4$ is a simple pole of *f(z)*.

Example 2. What type of singularity have the following functions:

(i) $\dfrac{1}{1-e^z}$ $\qquad$ (ii) $\dfrac{e^{2z}}{(z-1)^4}$ $\qquad$ (iii) $\dfrac{e^{1/z}}{z^2}$

Solution:

(i) Poles of $f(z) = 1/(1 - e^z)$ are found by equating to zero $1 - e^z = 0$ or $e^z = 1 = e^{2n\pi i}$

$\therefore\ z = 2n\pi i\ (n = 0, \pm 1, \pm 2, \ldots)$

Clearly f(z) has a simple pole at $z = 2\pi i$.

(ii) $\dfrac{e^{2z}}{(z-1)^4} = \dfrac{e^{2(t+1)}}{t^4} = \dfrac{e^2}{t^4}\cdot e^{2t}$ where $t = z - 1$

$$= \frac{e^2}{t^4}\left\{1 + \frac{2t}{1!} + \frac{(2t)^2}{2!} + \frac{(2t)^3}{3!} + \frac{(2t)^4}{4!} + \frac{(2t)^5}{5!} + ...\right\}$$

$$= e^2\left\{\frac{1}{t^4}+\frac{2}{t^3}+\frac{2}{t^2}+\frac{4}{3t}+\frac{2}{3}+\frac{4t}{15}+\ldots\right\}$$

$$= e^2\left\{\frac{1}{(z-1)^4}+\frac{2}{(z-1)^3}+\frac{2}{(z-1)^2}+\frac{4}{3(z-1)}+\frac{2}{3}+\frac{4(z-1)}{15}+\ldots\right\}$$

Since there are finite (4) number of terms containing negative powers of (z-1),

$\therefore$ Z = 1 is a pole of 4th order.

(ii) $$f(z)=\frac{e^{1/z}}{z^2}=\frac{1}{z^2}\left\{1+\frac{1}{1!\,z}+\frac{1}{2!\,z^2}+\frac{1}{3!\,z^3}+\ldots\right\}=z^{-2}+z^{-3}+\frac{z^{-4}}{2}+\ldots$$

Since there are infinite number of terms in the negative powers of z, therefore *f(z)* has an essential singularity at *z =0.*

2.15. Residues: Let the Laurent expansion of a function $f(z)$ around an isolated singularity $z = a$ be

$$f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \cdots + \frac{b_1}{z-a} + \frac{b_2}{(z-a)^2} + \cdots$$

In the expansion, b_1, the coefficient of $\frac{1}{z-a}$ is called the ***residue*** of $f(z)$ at the point $z = a$. This is written as

$$b_1 = Res. f(z)_{z=a}$$

But, we know already that

$$b_1 = \frac{1}{2\pi i}\int_C f(z)\, dz$$

where C is a curve surrounding $z = a$.

$$\text{Hence } \frac{1}{2\pi i}\int_C f(z)\, dz = Res. f(z)_{z=a}$$

$$\text{or} \int_C f(z)\, dz = 2\pi i\,\{Res. f(z)_{z=a}\} \qquad (1)$$

It may sometimes happen that Laurent expansion for $f(z)$ around $z = a$ can be easily got by algebraic manipulation. In that case b_1 can be calculated and hence we may compute the integral

$$\int_C f(z)\, dz \text{ using the formula(1).}$$

2.16. Cauchy's Residue Theorem: If C is a closed curve and $f(z)$ is analytic within and on C except at a finite number of singular points within C, then

$$\int_C f(z)\,dz = 2\pi i(r_1 + r_2 + \cdots + r_n)$$

where $r_1, r_2, \dots, r_n$ are the residues of the function $f(z)$ at the singular points.

Proof: Let $f(z)$ be a function analytic within and on the boundary of a region R at all points except at the points $z_1,\ z_2, \dots,\ z_n$. Around each singular point, we can draw a circle so small that it encloses no other singular point. Then these circles, together with the curve C from the boundary of a multiply connected region in which $f(z)$ is everywhere analytic. Applying Cauchy's integral theorem to the function $f(z)$ extended to the multiply connected region, we have

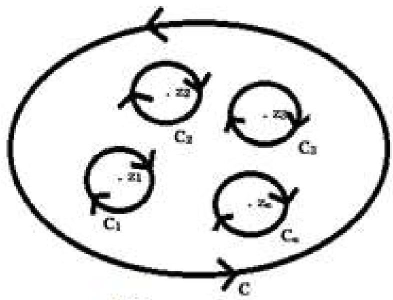


Figure 1

$$\frac{1}{2\pi i}\int_C f(z)\,dz + \frac{1}{2\pi i}\int_{C_1} f(z)\,dz + \frac{1}{2\pi i}\int_{C_2} f(z)\,dz + \cdots + \frac{1}{2\pi i}\int_{C_n} f(z)\,dz = 0 \qquad (1)$$

In the integration in the combined contour, it is clear that the integration round C is in the anticlockwise direction and the integrations around $C_1,\ C_2, \dots,\ C_n$ are in the clockwise direction. Hence in (1), we can reverse the direction of integration around $C_1,\ C_2, \dots,\ C_n$ and change the sign of these integrals. Then we have

$$\frac{1}{2\pi i}\int_C f(z)\,dz = \frac{1}{2\pi i}\int_{C_1} f(z)\,dz + \frac{1}{2\pi i}\int_{C_2} f(z)\,dz + \cdots + \frac{1}{2\pi i}\int_{C_n} f(z)\,dz \qquad (2)$$

where are the integrals are taken in the anticlockwise direction.

$$\text{But}\frac{1}{2\pi i}\int_{C_1} f(z)\,dz = Res.f(z)_{z=z_1} = r_1$$

$$\text{Similarly}\frac{1}{2\pi i}\int_{C_2} f(z)\,dz = Res.f(z)_{z=z_2} = r_2\text{and so on.}$$

Hence (2) gives

$$\frac{1}{2\pi i}\int_C f(z)\,dz = r_1 + r_2 + \cdots + r_n$$

$$i.e., \int_C f(z)\,dz = 2\pi i(r_1 + r_2 + \cdots + r_n)$$

EXAMPLES

EX. 1. Find the poles and residues of $\frac{z}{z^2 - 3z + 2}$.

Solution: Let $f(z) = \frac{z}{z^2 - 3z + 2} = \frac{z}{(z-1)(z-2)}$

Hence $z = 1$ and $z = 2$ are two simple poles.

To find the residue at $z = 1$, we expand the function in a Laurent series in powers of $z - 1$. We can split $f(z)$ into partial fractions.

We have

$$f(z) = -\frac{1}{z-1} + \frac{2}{z-2}$$

To expand in powers of $z - 1$, we write

$$f(z) = -\frac{1}{z-1} + \frac{2}{z-1-1}$$

$$= -\frac{1}{u} + \frac{2}{u-1} \text{ where } u = z - 1$$

$$= -\frac{1}{u} - \frac{2}{1-u}$$

$$= -\frac{1}{u} - 2(1-u)^{-1}$$

$$= -\frac{1}{u} - 2(1 + u + u^2 + u^3 + \cdots)$$

$$= -\frac{1}{z-1} - 2[1 + (z-1) + (z-1)^2 + \cdots]$$

Coefficient of $\frac{1}{z-1} = -1$ and hence residue at $z = 1$ is -1.

To find the residue at $z = 2$, we expand $f(z)$ in a Laurent's series in powers of $z - 2$.

We have

$$f(z) = -\frac{1}{z-1} + \frac{2}{z-2} = \frac{2}{z-2} - \frac{1}{z-2+1}$$

$$= \frac{2}{u} - \frac{1}{1+u} \text{ where } u = z - 2$$

$$= \frac{2}{u} - (1+u)^{-1}$$

$$= \frac{2}{u} - (1 - u + u^2 - u^3 + \cdots)$$

$$= \frac{2}{z-2} - [1 - (z-2) + (z-2)^2 - (z-2)^3 + \cdots]$$

Coefficient of $\frac{1}{z-2} = 2$ and hence residue at $z = 2$ is 2.

EX. 2. Obtain the Laurent expansion of the function $\frac{e^z}{(z-1)^2}$,

in the neighbourhood of its singular point and hence find its residue.

Solution: Let $f(z) = \frac{e^z}{(z-1)^2}$

$z = 1$ is a pole of the second order for the given function.

$$f(z) = \frac{e^z}{(z-1)^2} = \frac{e^{z-1+1}}{(z-1)^2}$$

$$= \frac{e^{u+1}}{u^2} \text{ putting } z - 1 = u$$

$$= \frac{e^u . e}{u^2}$$

$$= \frac{e}{u^2}\left(1 + \frac{u}{1} + \frac{u^2}{2!} + \frac{u^3}{3!} + \cdots\right)$$

$$= \frac{e}{u^2} + \frac{e}{u} + \frac{e}{u^2}\left(\frac{u^2}{2!} + \frac{u^3}{3!} + \cdots\right)$$

Coefficient of $\frac{1}{u}$ $i.e.,$ of $\frac{1}{z-1}$ $is = e$.

Therefore the required residue= e.

EX.3. Find the poles and residues of $\frac{1}{z^2-1}$.

Solution: Let $f(z) = \frac{1}{z^2-1} = \frac{1}{(z-1)(z+1)} = \frac{1}{2}\left(\frac{1}{z-1} - \frac{1}{z+1}\right)$, using partial fractions.

Poles of $f(z)$ are given by $(z-1)(z+1) = 0$, $i.e., z = \pm 1$.

These are simple poles.

To find the residue at $z = 1$, we expand the function in a Laurent series in powers of $z - 1$.

To expand in powers of $z - 1$, we write

$$f(z) = \frac{1}{2}\left(\frac{1}{z-1} - \frac{1}{z-1+2}\right) = \frac{1}{2}\left(\frac{1}{u} - \frac{1}{u+2}\right) \text{where } u = z - 1$$

$$= \frac{1}{2u} - \frac{1}{4}\left(\frac{1}{1+\frac{u}{2}}\right) = \frac{1}{2u} - \frac{1}{4}\left(1+\frac{u}{2}\right)^{-1}$$

$$= \frac{1}{2u} - \frac{1}{4}\left(1 - \frac{u}{2} + \frac{u^2}{4} - \frac{u^3}{8} + \cdots\right)$$

$$= \frac{1}{2u} - \frac{1}{4} + \frac{u}{8} - \frac{u^2}{16} - \frac{u^3}{32} - \cdots$$

$$= -\frac{1}{4} + \frac{1}{2(z-1)} + \frac{z-1}{8} - \frac{(z-1)^2}{16} - \cdots$$

Coefficient of $\frac{1}{z-1} = \frac{1}{2}$ and hence residue at $z = 1$ is $\frac{1}{2}$.

To find the residue at $z = -1$, we expand $f(z)$ in a Laurent series in powers of $z + 1$, we have

$$f(z) = \frac{1}{2}\left(\frac{1}{z+1-2} - \frac{1}{z+1}\right) = \frac{1}{2}\left(\frac{1}{u-2} - \frac{1}{u}\right) \text{ where } u = z+1$$

$$= -\frac{1}{4}\left(\frac{1}{1-\frac{u}{2}}\right) - \frac{1}{2u}$$

$$= -\frac{1}{4}\left(1-\frac{u}{2}\right)^{-1} - \frac{1}{2u}$$

$$= -\frac{1}{4}\left(1 + \frac{u}{2} + \frac{u^2}{4} + \frac{u^3}{8} + \cdots\right) - \frac{1}{2u}$$

$$= -\frac{1}{4}\left(1 + \frac{z+1}{2} + \frac{(z+1)^2}{4} + \frac{(z+1)^3}{8} + \cdots\right) - \frac{1}{2(z+1)}$$

Coefficient of $\frac{1}{z+1} = -\frac{1}{2}$.

Therefore residue at $z = -1$ is $-\frac{1}{2}$.

EX.4. Expand $f(z) = \frac{z}{(z+1)(z+2)}$ as a Laurent series about $z = -2$ and hence find the residue at that point.

Solution: Given $f(z) = \dfrac{z}{(z+1)(z+2)}$

Here $z = -1$ and $z = -2$ are two simple poles.

To find the residue at $z = -2$, we expand the function in powers of $z + 2$.

We can split $f(z)$ into partial fractions.

We have $f(z) = \dfrac{z}{(z+1)(z+2)} = \dfrac{2}{z+2} - \dfrac{1}{z+1}$

To expand in powers of $z+2$, we write

$$f(z) = \frac{2}{z+2} - \frac{1}{(z+2)-1}$$

$$= \frac{2}{z+2} + \frac{1}{1-(z+2)}$$

$$= \frac{2}{u} + \frac{1}{1-u} = \frac{2}{u} + (1-u)^{-1}, \text{where } u = z+1$$

$$= \frac{2}{u} + (1 + u + u^2 + u^3 + \cdots \infty)$$

$$= \frac{2}{z+2} + 1 + (z+2) + (z+2)^2 + \cdots$$

Coefficient of $\frac{1}{z+2} = 2$ and hence residue at $z = -2$ is 2.

2.17. **Evaluation of residues**: The calculation of residues by the use of series expansion, as illustrated in worked examples 1 and 2 is often tedious. Hence alternative procedures are available to determine residues. We now consider them.

(i) Let $f(z)$ have a simple or first order pole at $z = a$.

We can then write

$$f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \cdots + \frac{b_1}{z-a} \qquad (1)$$

By definition, $b_1 = \{Res. f(z)\}_{z=1}$

Multiplying (1) by $z-a$, we have,

$$(z-a)f(z) = a_0(z-a) + a_1(z-a)^2 + \cdots + b_1$$

Taking limit of both sides as $z \to a$, we have

$$\lim_{z\to a}(z-a)f(z) = b_1$$

$$\text{Therefore}\{Res. f(z)\}_{z=1} = \lim_{z\to a}(z-a)f(z)$$

(ii) Often, it will be necessary to calculate the residues of a function $f(z)$ of the form $f(z) = \frac{\emptyset(z)}{\Psi(a)}$,

where $\Psi(z)$ has simple zeroes and hence $f(z)$ simple poles.

Let $z = a$ be a simple pole of $f(z)$. Then $\Psi(a)$ must be =0.

$$\{Res. f(z)\}_{z=a} = \lim_{z\to a}(z-a)\frac{\emptyset(z)}{\Psi(z)}$$

$$= \lim_{z\to a}(z-a)\frac{[\emptyset(a) + (z-a)\emptyset'(a) + \cdots]}{[\Psi(a) + (z-a)\Psi'(a) + \cdots]} \text{ (by Taylor'stheorem)}$$

$$= \lim_{z \to a} \frac{(z-a)\emptyset(a) + (z-a)^2 \emptyset'(a) + \cdots}{(z-a)\Psi'(a) + \cdots} \text{ (since } \Psi(a) = 0)$$

$$= \frac{\emptyset(a)}{\Psi'(a)}$$

(iii) Suppose $f(z)$ has a second order pole $z = a$. Then we have

$$f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \cdots + \frac{b_1}{z-a} + \frac{b_2}{(z-a)^2} \qquad (1)$$

b_1 is the residue at $z = a$ and it has to be found.

Multiply both sides of (1) by $(z-a)^2$, we get

$$(z-a)^2 f(z) = a_0(z-a)^2 + a_1(z-a)^3 + a_2(z-a)^4 + \cdots + b_1(z-a) + b_2 \qquad (2)$$

Differentiating both sides of (2) with respect to z, we get

$$\frac{d}{dz}[(z-a)^2 f(z)] = 2a_0(z-a) + 3a_1(z-a)^2 + 4a_2(z-a)^3 + \cdots + b_1 \qquad (3)$$

Take the limit of both sides of (3), as $z \to a$.

Then$b_1 = \lim_{z \to a} \frac{d}{dz}[(z-a)^2 f(z)]$

Similarly let $f(z)$ has a pole of order m at $z = a$. Then we have

$$f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \cdots + \frac{b_1}{z-a} + \frac{b_2}{(z-a)^2} + \cdots + \frac{b_m}{(z-a)^m} \qquad (4)$$

Multiplying both sides of (4) by $(z-a)^m$, we have

$$(z-a)^m f(z) = a_0(z-a)^m + a_1(z-a)^{m+1} + a_2(z-a)^{m+2} + \cdots$$
$$+ b_1(z-a)^{m-1} + b_2(z-a)^{m-2} + \cdots + b_m \qquad (5)$$

Differentiate both sides of (5) with respect to z, $m-1$times, we get

$$\frac{d^{m-1}}{dz^{m-1}}[(z-a)^m f(z)] = b_1.(m-1)! + terms\ containing\ (z-a) \qquad (6)$$

Take the limit of both sides of (6), as $z \to a$, we get

$$\lim_{z \to a} \frac{d^{m-1}}{dz^{m-1}}[(z-a)^m f(z)] = b_1.(m-1)!$$

$$\text{or } b_1 = \frac{1}{(m-1)!} \lim_{z \to a} \frac{d^{m-1}}{dz^{m-1}}[(z-a)^m f(z)]$$

EX. 5. Find the poles and residues of $\frac{z}{z^2 - 3z + 2}$.

Solution: Let $f(z) = \frac{z}{z^2 - 3z + 2} = \frac{z}{(z-1)(z-2)}$

Here $z = 1$ and $z = 2$ are simple poles of $f(z)$.

Then $\{Res. f(z)\}_{z=1} = \lim_{z\to 1}(z-1).f(z)$

$$= \lim_{z\to 1}(z-1).\frac{z}{(z-1)(z-2)}$$

$$= \lim_{z\to 1}\frac{z}{(z-2)} = -1$$

$\{Res. f(z)\}_{z=2} = \lim_{z\to 2}(z-2).f(z)$

$$= \lim_{z\to 2}(z-2).\frac{z}{(z-1)(z-2)}$$

$$= \lim_{z\to 2}\frac{z}{(z-1)} = 2$$

EX. 6. Find the residue of $\frac{ze^z}{(z-a)^3}$ at $z = a$.

Solution: Let $f(z) = \frac{ze^z}{(z-a)^3}$

Here $z = a$ is a pole of order 3 for $f(z)$.

Then $\{Res. f(z)\}_{z=a} = \frac{1}{2!}\lim_{z\to a}\frac{d^2}{dz^2}(z-a)^3.f(z)$

$$= \frac{1}{2}\lim_{z\to a}\frac{d^2}{dz^2}(ze^z)$$

$$= \frac{1}{2}\lim_{z\to a}\frac{d}{dz}(ze^z + e^z)$$

$$= \frac{1}{2}\lim_{z\to a}(ze^z + 2e^z) == \frac{1}{2}e^a(a+2)$$

EX.7. Evaluate the residues at the poles of the function $f(z) = \frac{z^2-2z}{(z+1)^2(z^2+4)}$

Solution: Given that $f(z) = \frac{z^2-2z}{(z+1)^2(z^2+4)} = \frac{z^2-2z}{(z+1)^2(z-2i)(z+2i)}$

For the given function, $z = -1$ is a pole of the second order and $z = 2i, -2i$ are two simple poles.

Then $\{Res. f(z)\}_{z=-1} = \lim_{z\to -1}\frac{d}{dz}(z+1)^2.f(z)$

$$= \lim_{z\to -1}\frac{d}{dz}(z+1)^2.\frac{z^2-2z}{(z+1)^2(z^2+4)}$$

$$= \lim_{z\to -1}\frac{d}{dz}\frac{z^2-2z}{(z^2+4)}$$

$$= \lim_{z \to -1} \frac{(z^2+4)(2z-2)-(z^2-2z)\,2z}{(z^2+4)^2} \quad = -\frac{14}{15}$$

$$\{Res. f(z)\}_{z=2i} = \lim_{z \to 2i} (z-2i) . f(z)$$

$$= \lim_{z \to 2i} (z-2i) . \frac{z^2-2z}{(z+1)^2(z-2i)(z+2i)}$$

$$= \lim_{z \to 2i} \frac{z^2-2z}{(z+1)^2(z+2i)}$$

$$= \lim_{z \to 2i} \frac{z^2-2z}{(z+1)^2(z+2i)} = \frac{7+i}{25}$$

$$\{Res. f(z)\}_{z=-2i} = \lim_{z \to -2i} (z+2i) . f(z)$$

$$= \lim_{z \to -2i} (z+2i) . \frac{z^2-2z}{(z+1)^2(z-2i)(z+2i)}$$

$$= \lim_{z \to -2i} \frac{z^2-2z}{(z+1)^2(z-2i)}$$

$$= \lim_{z \to -2i} \frac{z^2-2z}{(z+1)^2(z-2i)} \quad = \frac{7-i}{25}$$

EX. 8. Determine the poles of the function $(i)\ \frac{z}{cos\ z}$ $\quad (ii)\ cot\ z$.

Solution: (i) The poles of $f(z) = \frac{z}{cos\ z}$ are given by

$$cos\ z = 0$$

$i.e., z = (2n+1)\frac{\pi}{2}, n$ being zero or an integer

$$i.e., z = (2n+1)\frac{\pi}{2}, n = 0, \pm 1, \pm 2, \ldots$$

Hence these are simples of $f(z)$.

(ii) The poles of $f(z) = cot\ z$ are given by $sin\ z = 0$

$$i.e., z = n\pi, n = 0, \pm 1, \pm 2, \ldots$$

Which are simples of $f(z)$.

We know that residue of $f(z) = \frac{\phi(z)}{\Psi(z)}$ at $z = z_0$ is $\frac{\phi(z_0)}{\Psi'(z_0)}$

$\therefore$ Residue of $f(z)$ at $z = n\pi\ is = \left(\frac{cos\ z}{cos\ z}\right)_{z=n\pi} = \frac{cos\ n\pi}{cos\ n\pi} = 1$

EX. 9. Find the poles of the function $f(z) = \frac{1}{(z+1)(z+3)}$ and the residues at these poles.

Solution: The given function $f(z)$ has two simple poles at $z = -1$ and $z = -3$.

$\therefore$ Residue of $f(z)$ at $z=-1$ is $\lim_{z\to -1}\{(z+1)f(z)\} = \lim_{z\to -1}\frac{1}{z+3} = \frac{1}{2}$

Also the Residue of $f(z)$ at $z=-3$ is $\lim_{z\to -3}\{(z+3)f(z)\} = \lim_{z\to -3}\frac{1}{z+1} = -\frac{1}{2}$

EX. 10. Find the poles of the function $f(z) = \dfrac{z^2}{(z-1)(z-2)^2}$ and the residues at these poles.

Solution: The given function $f(z)$ has two simple pole at $z=1$ and another pole of order 2 at $z=2$.

$\therefore$ Residue of $f(z)$ at $z=-1$ is

$$\lim_{z\to 1}\{(z-1)f(z)\} = \lim_{z\to 1}\frac{z^2}{(z-2)^2} = \frac{1}{1} = 1$$

Also the Residue of $f(z)$ at $z=2$ is

$$\lim_{z\to 2}\frac{d}{dz}\{(z-2)^2 f(z)\} = \lim_{z\to 2}\frac{d}{dz}\left(\frac{z^2}{z-1}\right)$$

$$= \lim_{z\to 2}\frac{(z-1)(2z)-z^2.1}{(z-1)^2}$$

$$= \lim_{z\to 2}\frac{z^2-2z}{(z-1)^2} = 0$$

EX. 11. Determine the poles of the function $f(z) = \dfrac{z^2}{(z+2)(z-1)^2}$ and the residues at each pole .

Solution: $z=1$ and $z=-2$ are the zeros of denominator of order 2 and 1 respectively.

$\therefore$ $z=1$ is a pole of order 2 and $z=-2$ is pole of order 1 of $f(z)$.

$$[Res\ f(z)]_{z=-2} = \lim_{z\to -2}\{(z+2)f(z)\} = \lim_{z\to -3}\frac{z^2}{(z-1)^2} = \frac{4}{9}$$

$$[Res\ f(z)]_{z=1} = \lim_{z\to 1}\frac{d}{dz}\{(z-1)^2 f(z)\} = \lim_{z\to 1}\frac{d}{dz}\left(\frac{z^2}{z+2}\right) = \frac{5}{9}$$

EX. 12. Find the residue of $\dfrac{ze^z}{(z-1)^3}$ at its pole .

Solution: Let $f(z) = \dfrac{ze^z}{(z-1)^3}$

Poles of $f(z)$ are obtained by putting the denominator equal to zero.

$\therefore$ $z=1$ is a pole of $f(z)$ of order 3.

We know that if $f(z)$ has a pole of order m at $z = a$ then,

$$[Res\, f(z)]_{z=a} = \frac{1}{(m-1)!}\lim_{z\to a}\frac{d^{m-1}}{dz^{m-1}}\{(z-a)^m f(z)\}$$

Here $a = 1, m = 3$

$$\therefore \quad [Res\, f(z)]_{z=1} = \frac{1}{2!}\lim_{z\to 1}\frac{d^2}{dz^2}\{(z-1)^3 f(z)\}$$

$$= \frac{1}{2}\lim_{z\to 1}\frac{d^2}{dz^2}(ze^z) \quad = \frac{1}{2}\lim_{z\to 1}\frac{d}{dz}(ze^z + e^z)$$

$$= \frac{1}{2}\lim_{z\to 1}(ze^z + e^z + e^z) = \frac{3e}{2}$$

EX. 13. Find the residue at $z = 0$ of the function $f(z) = \dfrac{1+e^z}{sin\, z + z\, cos\, z}$.

Solution: The residue of $f(z)$ at $z = 0$ is

$$\lim_{z\to 0}\{zf(z)\} = \lim_{z\to 0} z.\frac{1+e^z}{sin\, z + z\, cos\, z} = \lim_{z\to 0} z.\frac{1+e^z}{z\left(\frac{sin\, z}{z} + \, cos\, z\right)}$$

$$= \lim_{z\to 0}\frac{1+e^z}{\left(\frac{sin\, z}{z} + \, cos\, z\right)} = \frac{2}{2} = 1\left(\because \lim_{z\to 0}\frac{sin\, z}{z} = 1\right)$$

An Alternate Method:

$$\text{Residue at } z = 0 \text{ is} = \lim_{z\to 0}\{(z-0)f(z)\} = \lim_{z\to 0}\frac{z(1+e^z)}{sin\, z + z\, cos\, z}\left(=\frac{0}{0}\right)$$

$$= \lim_{z\to 0}\frac{ze^z + (1+e^z)(1)}{cos\, z + z\,(-sin\, z) + cos\, z} \text{ (using L'Hospital rule)}$$

$$= \frac{0+1+1}{1+0+1} = 1$$

EX. 14. Find the residues of the function $f(z) = \dfrac{1-e^{2z}}{z^4}$ at the poles.

Solution: $z = 0$ is the singular point of $f(z)$

$$\text{Expanding } f(z) = \frac{1-e^{2z}}{z^4} = \frac{1-\left[1+\frac{2z}{1!}+\frac{4z^2}{2!}+\frac{8z^3}{3!}+\cdots\right]}{z^4}$$

$$= -\left[\frac{2}{z^3}+\frac{2}{z^2}+\frac{4}{3}.\frac{1}{z}+\frac{2}{3}+\frac{4}{15}z+\cdots\right]$$

$z = 0$ is a pole of order 3, because $\frac{1}{z^3}$ is the highest negative power of $(z-0)$.

The residue of $f(z)$ at $z = 0$ is $-\frac{4}{3}$.

EX. 15. Find the residue of $f(z) = \dfrac{z^3}{(z-1)^4(z-2)(z-3)}$ at $z = 1$.

Solution: Let $\phi(z) = \dfrac{z^3}{(z-2)(z-3)}$, so that $f(z) = \dfrac{\phi(z)}{(z-1)^4}$.

Here $z = 1$ is a pole of order 4.

∴ Residue of $f(z)$ at $z = 1$ is

∴ Residue of $f(z)$ at $z = 1$ is $= [Res\, f(z)]_{z=1} = \dfrac{\phi'''(1)}{3!}$

But $\phi(z) = \dfrac{z^3}{(z-2)(z-3)} = z + 5 - \dfrac{8}{z-2} + \dfrac{27}{z-3}$ (on resolving partial fractions)

$$\phi'(z) = 1 + \frac{8}{(z-2)^2} - \frac{27}{(z-3)^2}$$

$$\phi''(z) = -\frac{16}{(z-2)^3} + \frac{54}{(z-3)^3}$$

$$\phi'''(z) = \frac{48}{(z-2)^4} - \frac{162}{(z-3)^4}$$

$$\therefore\ \phi'''(1) = 48 - \frac{162}{16} = \frac{303}{8}$$

$$\therefore\ [Res\, f(z)]_{z=1} = \frac{\phi'''(1)}{3!} = \frac{1}{3!}.\frac{303}{8} = \frac{101}{16}$$

EX.16. Find the residue of

$f(z) = \dfrac{z^2}{z^4+1}$ at these singular points which lie inside the circle $|z| = 2$.

Solution: Let $f(z) = \dfrac{z^2}{z^4+1}$

Poles of $f(z)$ are obtained by putting the denominator equal to zero.

$i.e.,\quad z^4 + 1 = 0\ or\ z^4 = -1$

or $z = (-1)^{1/4} = (cos\, \pi + i\, sin\, \pi)^{1/4}$

$$= cos\left(\frac{2n\pi + \pi}{4}\right) + i\, sin\left(\frac{2n\pi + \pi}{4}\right)$$

∴ The four values of z are

$cos\,\dfrac{\pi}{4} + i\, sin\,\dfrac{\pi}{4}, cos\,\dfrac{3\pi}{4} + i\, sin\,\dfrac{3\pi}{4}, cos\,\dfrac{5\pi}{4} + i\, sin\,\dfrac{5\pi}{4}$ and $cos\,\dfrac{7\pi}{4} + i\, sin\,\dfrac{7\pi}{4}$

$i.e., \dfrac{1}{\sqrt{2}} + i\dfrac{1}{\sqrt{2}},\quad -\dfrac{1}{\sqrt{2}} + i\dfrac{1}{\sqrt{2}},\quad -\dfrac{1}{\sqrt{2}} - i\dfrac{1}{\sqrt{2}}$ and $\dfrac{1}{\sqrt{2}} - i\dfrac{1}{\sqrt{2}}$

or $\quad \dfrac{1+i}{\sqrt{2}}, \dfrac{-1+i}{\sqrt{2}}, \dfrac{-1-i}{\sqrt{2}}$ and $\dfrac{1-i}{\sqrt{2}}$

Hence the simple poles of $f(z)$ are $\frac{\pm 1 \pm i}{\sqrt{2}}$ and all these lies within the circle $|z| = 2$ with centre 0 and radius 2.

Now let $f(z) = \dfrac{z^2}{z^4+1} = \dfrac{\phi(z)}{\Psi(z)}$

$$\therefore \quad [Res\, f(z)]_{z=\frac{1+i}{\sqrt{2}}} = \frac{\phi\left(\frac{1+i}{\sqrt{2}}\right)}{\Psi'\left(\frac{1+i}{\sqrt{2}}\right)} = \frac{i}{4i\left(\frac{1+i}{\sqrt{2}}\right)} = \frac{1}{2\sqrt{2}(1+i)}$$

$$= \frac{1-i}{4\sqrt{2}} \quad \left[\because [Res\, f(z)]_{z=z_0} = \frac{\phi(z_0)}{\Psi(z_0)}\right]$$

$$[Res\, f(z)]_{z=\frac{-1+i}{\sqrt{2}}} = \frac{\phi\left(\frac{-1+i}{\sqrt{2}}\right)}{\Psi'\left(\frac{-1+i}{\sqrt{2}}\right)} = \frac{-i}{-4i\left(\frac{-1+i}{\sqrt{2}}\right)} = \frac{1}{2\sqrt{2}(-1+i)} = \frac{-1-i}{4\sqrt{2}}$$

$$[Res\, f(z)]_{z=\frac{-1-i}{\sqrt{2}}} = \frac{\phi\left(\frac{-1-i}{\sqrt{2}}\right)}{\Psi'\left(\frac{-1-i}{\sqrt{2}}\right)} = \frac{-i}{-4i\left(\frac{1+i}{\sqrt{2}}\right)} = \frac{-1}{2\sqrt{2}(1+i)} = \frac{-1-i}{4\sqrt{2}}$$

$$[Res\, f(z)]_{z=\frac{1-i}{\sqrt{2}}} = \frac{\phi\left(\frac{1-i}{\sqrt{2}}\right)}{\Psi'\left(\frac{-1-i}{\sqrt{2}}\right)} = \frac{-i}{-4i\left(\frac{1-i}{\sqrt{2}}\right)} = \frac{1}{2\sqrt{2}(1-i)} = \frac{1+i}{4\sqrt{2}}$$

EX. 17. Find the residue of $(i)\dfrac{z^2-2z}{(z+1)^2(z^2+1)}$ $\quad (ii)\, tan\, z$ at each pole.

Solution: (i) Let $f(z) = \dfrac{z^2-2z}{(z+1)^2(z^2+1)}$

$\therefore$ Poles of $f(z)$ are $-1, i$ and $-\, i$.

Observe that -1 is a pole of order two and the poles $\pm i$ are or order one.

$$\therefore \quad [Res\, f(z)]_{z=-1} = \frac{1}{(2-1)!}\lim_{z\to -1}\left[\frac{d}{dz}(z+1)^2 f(z)\right] = \lim_{z\to -1}\left[\frac{d}{dz}\left(\frac{z^2-2z}{z^2+1}\right)\right]$$

$$= \lim_{z\to -1}\left[\frac{(z^2+1)(2z-2)-(z^2-2z)(2z)}{(z^2+1)^2}\right] = -\frac{1}{2}$$

$$[Res\, f(z)]_{z=i} = \operatorname*{liim}_{z\to i}[(z-i)f(z)]$$

$$= \operatorname*{liim}_{z\to i}\left[\frac{z^2-2z}{(z+1)^2(z+i)}\right] = \frac{i^2-2i}{(i+1)^2(i+i)} = \frac{1+2i}{4}$$

$$[Res\, f(z)]_{z=-i} = \lim_{z\to -i}[(z+i)f(z)]$$

$$= \lim_{z\to -i}\left[\frac{z^2-2z}{(z+1)^2(z-i)}\right] = \frac{(-i)^2-2(-i)}{(-i+1)^2(-2i)} = \frac{1-2i}{4}$$

(ii) Let $f(z) = tan\, z = \dfrac{sin\, z}{cos\, z}$

∴ Poles of $f(z)$ are given by $cos\, z = 0$

$i.e.,\quad z = (2n+1)\dfrac{\pi}{2}, where\ n = 0, \pm1, \pm2, \pm3, \ldots$

All these poles are simple poles of $f(z)$.

Denoting each pole by $'a'$, we have

$$[Res\, f(z)]_{z=a} = \lim_{z\to a}[(z-a)f(z)]$$

$$= \lim_{z\to a}\frac{(z-a)f(z)}{cos\, z} \quad \left(=\frac{0}{0}\right)$$

$$= \lim_{z\to a}\frac{(z-a)cos\, z + sin\, z}{-sin\, z} \quad \text{(L − Hospital Rule)}$$

$$= -1$$

Hence residue of $f(z)$ at each of the poles is -1.

EX. 18. Find the poles and residues of $\dfrac{3z+1}{(z+1)(2z-1)}$.

Solution: Let $f(z) = \dfrac{3z+1}{(z+1)(2z-1)}$

The poles of $f(z)$ are given by $(z+1)(2z-1) = 0$

$i.e.,\quad z+1 = 0 \quad or \quad 2z-1 = 0 \quad i.e., z = -1, \dfrac{1}{2}$

∴ $f(z)$ has two simple poles at $z = -1$ and $\frac{1}{2}$.

Residue at $z = -1$ is given by

$$[Res\, f(z)]_{z=-1} = \lim_{z\to -1}(z+1)f(z)$$

$$= \lim_{z\to -1}\frac{3z+1}{2z-1} = \frac{2}{3}$$

Residue at $z = \frac{1}{2}$ is given by

$$[Res\, f(z)]_{z=\frac{1}{2}} = \lim_{z\to \frac{1}{2}}\left(z-\frac{1}{2}\right)f(z)$$

$$= \frac{1}{2}\lim_{z\to \frac{1}{2}}(2z-1)f(z)$$

$$= \frac{1}{2}\lim_{z\to\frac{1}{2}}\frac{3z+1}{z+1} = \frac{1}{2}\left(\frac{\frac{3}{2}+1}{\frac{1}{2}+1}\right) = \frac{5}{6}$$

EX.19. Find the poles of $f(z)$ and the residues of the poles which lie on imaginary axis if

$$f(z) = \frac{z^2+2z}{(z+1)^2(z^2+4)}$$

Solution: (i) we have $f(z) = \dfrac{z^2+2z}{(z+1)^2(z^2+4)} = \dfrac{z^2+2z}{(z+1)^2(z+2i)(z-2i)}$

Poles of $f(z)$ are obtained by putting the denominator equal to zero.

∴ Poles of $f(z)$ are $z = -1, -2i, 2i$

Obviously $z = -1$ is a double pole and $z = \pm 2i$ are simple poles.

Now we have to calculate the residues at $z = \pm 2i$.

To calculate residue of $f(z)$ at $z = 2i$

$$[Res\, f(z)]_{z=2i} = \lim_{z\to 2i}(z-2i)f(z)$$

$$= \lim_{z\to 2i}\left[\frac{z^2+2z}{(z+1)^2(z+2i)}\right] = \frac{-4+4i}{(2i+1)^2(4i)}$$

$$= \frac{i-1}{i(-3+4i)} = \frac{1-i}{4+3i} = \frac{1-7i}{25}$$

To calculate residue of $f(z)$ at $z = -2i$

$$[Res\, f(z)]_{z=-2i} = \lim_{z\to -2i}(z+2i)f(z)$$

$$= \lim_{z\to -2i}\left[\frac{z^2+2z}{(z+1)^2(z-2i)}\right] = \frac{-4-4i}{(2i+1)^2(-4i)}$$

$$= \frac{i+1}{i(-3+4i)} = \frac{-1-i}{4+3i} = -\frac{1}{25}(7+i)$$

EX.20. Find the poles and residues at each pole of $tanh\, z$.

Solution: Let $f(z) = tanh\, z = \dfrac{sinh\, z}{cosh\, z} = \dfrac{e^z - e^{-z}}{e^z + e^{-z}} = \dfrac{e^{2z}-1}{e^{2z}+1}$

Poles of $f(z)$ are given by $e^{2z} + 1 = 0$

$i.e., (e^z + i)(e^z - i) = 0$

or $e^z = i, -i$ $or\ e^z = e^{i\frac{\pi}{2}}, e^{-i\frac{\pi}{2}}$

or $z = i\dfrac{\pi}{2}, -i\dfrac{\pi}{2}$

∴ $z = \pm i\dfrac{\pi}{2}$ are simple poles of $f(z)$.

Now, let $f(z) = \dfrac{e^{2z}-1}{e^{2z}+1} = \dfrac{\phi(z)}{\Psi(z)}$

Residue of $f(z)$ $at\ z = i\dfrac{\pi}{2}\ is = \dfrac{\phi\left(i\frac{\pi}{2}\right)}{\Psi'\left(i\frac{\pi}{2}\right)}$

$$= \left[\frac{e^{2z}-1}{\frac{d}{dz}(e^{2z}+1)}\right]_{z=i\frac{\pi}{2}} = \left[\frac{e^{2z}-1}{2e^{2z}}\right]_{z=i\frac{\pi}{2}}$$

$$= \frac{1}{2}[1-e^{-2z}]_{z=i\frac{\pi}{2}} = \frac{1}{2}\left(1-e^{-i\pi}\right) = 1$$

Residue of $f(z)$ $at\ z = -i\dfrac{\pi}{2}\ is = \dfrac{\phi\left(-i\frac{\pi}{2}\right)}{\Psi'\left(-i\frac{\pi}{2}\right)}$

$$= \left[\frac{e^{2z}-1}{\frac{d}{dz}(e^{2z}+1)}\right]_{z=-i\frac{\pi}{2}} = \left[\frac{e^{2z}-1}{2e^{2z}}\right]_{z=-i\frac{\pi}{2}}$$

$$= \frac{1}{2}[1-e^{-2z}]_{z=-i\frac{\pi}{2}} = \frac{1}{2}\left(1-e^{i\pi}\right) = 1$$

EX. 21. Find the poles and residues at each pole of $f(z) = \dfrac{sin^2 z}{\left(z-\frac{\pi}{6}\right)^2}$

Solution: We have $f(z) = \dfrac{sin^2 z}{\left(z-\frac{\pi}{6}\right)^2}$

Poles of $f(z)$ are obtained by putting the denominator equal to zero.

∴ Poles of $f(z)$ are $z = \frac{\pi}{6}$ is a double pole.

To calculate residue of $f(z)$ at $z = \frac{\pi}{6}$

Residue of $f(z)$ at $z = \dfrac{\pi}{6}$ is $= \dfrac{1}{(2-1)!} \lim_{z\to\frac{\pi}{6}} \left[\dfrac{d}{dz}\left\{\left(z-\frac{\pi}{6}\right)^2 f(z)\right\}\right]$

$$= \lim_{z\to\frac{\pi}{6}} \left[\frac{d}{dz}(sin^2 z)\right] = \lim_{z\to\frac{\pi}{6}}[2\ sin\ z.\ cos\ z]$$

$$= \lim_{z\to\frac{\pi}{6}}(sin\ 2z) = sin\ \left(\frac{2\pi}{6}\right) = sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$$

EX. 22. Find the poles and residues at each pole of $\frac{\cot z \coth z}{z^3}$.

Solution: The poles of $\frac{\cot z \coth z}{z^3}$ are given by $z = 0$ which is a pole of order 3.

Now $\frac{\cot z \coth z}{z^3} = \frac{\cos z \cosh z}{z^3 \sin z \sinh z}$

$$= \frac{1}{z^3} \frac{\left[1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots\right]\left[1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots\right]}{\left[z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\right]\left[z + \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots\right]}$$

$$= \frac{1}{z^3}\left(\frac{1 + \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^2}{2!} - \frac{z^4}{(2!)^2} - \frac{z^6}{2!\,4!} + \frac{z^4}{4!} + \frac{z^6}{2!\,4!} + \cdots}{z^2 + \frac{z^4}{3!} + \frac{z^6}{5!} - \frac{z^4}{3!} - \frac{z^6}{(3!)^2} - \frac{z^8}{3!\,5!} + \frac{z^6}{5!} + \frac{z^8}{3!\,5!} \cdots}\right)$$

$$= \frac{1}{z^3}\left(\frac{1 + z^4\left(\frac{1}{12} - \frac{1}{4}\right) + \cdots}{z^2 + \left[\frac{2}{5!} - \frac{1}{(3!)^2}\right] z^6 + \cdots}\right)$$

$$= \frac{1}{z^3}\left(\frac{1 - \frac{1}{6}z^4 + \cdots}{z^2 - \frac{1}{90}z^6 + \cdots}\right) = \frac{1}{z^5}\left(\frac{1 - \frac{1}{6}z^4 + \cdots}{1 - \frac{1}{90}z^4 + \cdots}\right)$$

EX. 23. Evaluate $\int_C \frac{z-2}{z(z-1)} dz$, where C is the circle $|z| = 2$.

Solution: Let $f(z) = \frac{z-2}{z(z-1)}$

The given function has two simple poles at $z = 0$ and $z = 1$.

These lie within the circle $|z| = 2$.

By Cauchy's Residue theorem,

$$\int_C f(z)\, dz = 2\pi i \times \text{sum of the residues of } f(z) \text{ at the interior poles.}$$

Hence we have to calculate the residues at $z = 0$ and $z = 1$.

Residue at $z = 0$ is $= \lim_{z\to 0} z \,.\, f(z)$

$= \lim_{z\to 0} z \,.\, \frac{z-2}{z(z-1)} \quad = \lim_{z\to 0} \frac{z-2}{z-1} = 2$

Residue at $z = 1$ is $= \lim_{z\to 1} (z-1) \,.\, f(z)$

$= \lim_{z\to 1} (z-1) \,.\, \frac{z-2}{z(z-1)}$

$$= \lim_{z\to 1} \frac{z-2}{z} = -1$$

Hence $\int_C \frac{z-2}{z(z-1)} dz = 2\pi i \times (2-1) = 2\pi i$

EX. 24. Evaluate $\int_C \frac{4-3z}{z(z-1)(z-2)} dz$, where C is the circumference of the circle $|z| = \frac{3}{2}$.

Solution: Let $f(z) = \frac{4-3z}{z(z-1)(z-2)}$

The given function has three first order poles at $z = 0, z = 1$ and $z = 2$. Of these, only $z = 0$ and $z = 1$ lie within the circle $|z| = \frac{3}{2}$.

Hence it is enough if we calculate the residues at $z = 0$ and $z = 1$.

Residue of $f(z)$ at $z = 0$ is $= \lim_{z\to 0} z \,.\, f(z)$

$$= \lim_{z\to 0} z \,.\, \frac{4-3z}{z(z-1)(z-2)}$$

$$= \lim_{z\to 0} \frac{4-3z}{(z-1)(z-2)}$$

$$= \frac{4}{-1 \times -2} = 2$$

Residue of $f(z)$ at $z = 1$ is $= \lim_{z\to 1}(z-1) \,.\, f(z)$

$$= \lim_{z\to 1}(z-1) \,.\, \frac{4-3z}{z(z-1)(z-2)}$$

$$= \lim_{z\to 1} \frac{4-3z}{z(z-2)}$$

$$= \frac{1}{1(1-2)} = -1$$

Therefore by Cauchy's Residue theorem, the value of the given integral

$= 2\pi i \times$ sum of the residues of $f(z)$ at the interior poles

$= 2\pi i\,(2-1) = 2\pi i$

EX. 25. Evaluate $\int_C \tan z \, dz$, where C is the curve $|z| = 2$.

Solution: Let $f(z) = \tan z = \frac{\sin z}{\cos z}$

The poles of $f(z)$ are given by $cos\, z = 0$

$i.e., z = (2n+1)\frac{\pi}{2}, n$ being zero or an integer.

Of the many poles, $z = \frac{\pi}{2}$ and , $z = -\frac{\pi}{2}$ are the only poles lying inside the given contour $|z| = 2$. Hence it is enough if we calculate the corresponding residues.

$$\text{Residue of } f(z) \text{ at } z = -\frac{\pi}{2} \text{ is } = \lim_{z \to -\frac{\pi}{2}} \frac{sin\, z}{\frac{d}{dz}(cos\, z)}$$

$$= \lim_{z \to -\frac{\pi}{2}} \frac{sin\, z}{-sin\, z} = -1$$

$$\text{Similarly Residue of } f(z) \text{ at } z = \frac{\pi}{2} \text{ is } = \lim_{z \to \frac{\pi}{2}} \frac{sin\, z}{\frac{d}{dz}(cos\, z)}$$

$$= \lim_{z \to \frac{\pi}{2}} \frac{sin\, z}{-sin\, z} = -1$$

Therefore by Cauchy's Residue theorem, the value of the given integral

$= 2\pi i \times$ sum of the residues of $f(z)$at the interior poles

$= 2\pi i\,(-1-1) = -4\pi i$

EX. 26. Evaluate $\int \frac{dz}{(z^2+4)^2}$,around the closed contour$|z - i| = 2$.

Solution: Let $f(z) = \frac{1}{(z^2+4)^2} = \frac{1}{(z-2i)^2(z+2i)^2}$

and this has a pole of order 2 at each of the points $z = 2i$ and $z = -2i$.

$|z - i| = 2$ is a circle with centre at $\boldsymbol{i}$and radius 2 units.

The pole $z = 2i$ is inside this circle and $z = -2i$ is outside.

So it is enough if we find the residue at $z = 2i$.

Residue at $z = 2i$ is

$$= \lim_{z \to 2i} \frac{d}{dz}(z-2i)^2 \frac{1}{(z-2i)^2(z+2i)^2}$$

$$= \lim_{z \to 2i} \frac{d}{dz} \frac{1}{(z+2i)^2}$$

$$= \lim_{z \to 2i} \frac{-2}{(z+2i)^3}$$

$$= \frac{-2}{(4i)^3} = \frac{1}{32\, i}$$

EX. 27. Evaluate $\oint_c \frac{2z-1}{z(z+2)(2z+1)}dz$ where c is the circle $|z|=1$.

Solution: Here $f(z)=\frac{2z-1}{z(z+2)(2z+1)}$ has three simple poles at $z=0, z=-2$ and $z=-\frac{1}{2}$.

But the only poles $z=0$ and $z=-\frac{1}{2}$ lies inside the circle $|z|=1$.

Now, the residue of $f(z)$ at $z=0$ is

$$\therefore \quad [Res\, f(z)]_{z=0}=\lim_{z\to 0} z\, f(z)=\lim_{z\to 0}\frac{2z-1}{(z+2)(2z+1)}=-\frac{1}{2.1}=-\frac{1}{2}$$

Also the residue of $f(z)$ at $z=-\frac{1}{2}$ is

$$[Res\, f(z)]_{z=-\frac{1}{2}}=\lim_{z\to -\frac{1}{2}}(2z+1)\, f(z)=\lim_{z\to -\frac{1}{2}}\frac{2z-1}{z(z+2)}=-\frac{2}{\frac{-3}{4}}=\frac{8}{3}$$

$\therefore$ By Residue theorem, we have

$$\oint_c \frac{2z-1}{z(z+2)(2z+1)}dz=2\pi i\left(-\frac{1}{2}+\frac{8}{3}\right)=\frac{13}{3}\pi i$$

EX. 28. Evaluate $\oint_c \tan z\, dz$ where c is the circle $|z|=2$.

Solution: Here $f(z)=\tan z=\frac{\sin z}{\cos z}$

The poles of $f(z)$ are given by $\cos z=0$

$i.e., z=\pm(2n+1)\frac{\pi}{2}, n=0,1,2 \ldots.$

Out of these only $z=\pm\frac{\pi}{2}$ (± 1.570) lies inside c: $|z|=2$

$$\therefore \quad [Res\, f(z)]_{z=\pm\frac{\pi}{2}}=\lim_{z\to\pm\frac{\pi}{2}}\left(z\pm\frac{\pi}{2}\right)\frac{\sin z}{\cos z}=\lim_{z\to\pm\frac{\pi}{2}}\frac{\sin z+\left(z\pm\frac{\pi}{2}\right).\cos z}{-\sin z}=-1$$

By using residue theorem, $\oint_c \tan z\, dz=2\pi i\,(-1-1)=-4\pi i$

EX. 29. Evaluate $\oint_c \frac{1}{(z^2+4)^2}dz$ where c is the circle $|z-i|=2$.

Solution: Here $f(z)=\frac{1}{(z^2+4)^2}$ has double poles at $z=\pm 2i$. Of these poles only z

$=2i$

lies inside c.

Again $\oint_c \frac{1}{(z^2+4)^2}dz=\oint_c \frac{1}{(z+2i)^2(z-2i)^2}dz$

Since $\frac{1}{(z+2i)^2}$ is analytic in c, apply Cauchy's integral formula for derivatives

$$\therefore \quad \oint_c \frac{1}{(z^2+4)^2}dz=\frac{2\pi i}{1!}\frac{d}{dz}\left[\frac{1}{(z+2i)^2}\right]_{z=2i}=\frac{\pi}{16}$$

EX. 30. Evaluate $\int_c \frac{z\,cos\,z}{\left(z-\frac{\pi}{2}\right)^3}dz$ where c is the circle (i) $|z-1|=1$ $\quad(ii)$ $|z|=2$.

Solution: $(\,i)$ Let $f(z)=\frac{z\,cos\,z}{\left(z-\frac{\pi}{2}\right)^3}$

$z=\frac{\pi}{2}$ is a pole of order 3 of the function $f(z)$ and it lies within the given circle.

$$\therefore \quad [Res\,f(z)]_{z=\frac{\pi}{2}}=\frac{1}{(3-1)!}\lim_{z\to\frac{\pi}{2}}\left[\frac{d^{3-1}}{dz^{3-1}}\left\{\left(z-\frac{\pi}{2}\right)^3\ f(z)\right\}\right]$$

$$=\frac{1}{2!}\lim_{z\to\frac{\pi}{2}}\left[\frac{d^2}{dz^2}(z\,cos\,z)\right]$$

$$=\frac{1}{2}\lim_{z\to\frac{\pi}{2}}\left[\frac{d}{dz}(cos\,z-z\,sin\,z)\right]$$

$$=\frac{1}{2}\lim_{z\to\frac{\pi}{2}}(-z\,cos\,z-2\,sin\,z)=-1$$

$\therefore$ By Residue theorem,

$$\int_c \frac{z\,cos\,z}{\left(z-\frac{\pi}{2}\right)^3}dz=2\pi i\times \text{sum of the residues of } f(z) \text{ at the interior poles}$$

$$=2\pi i(-1)=-2\pi i$$

EX. 31. Evaluate $\int_c \frac{coth\, z}{z-i} dz$ where c is $|z| = 2$.

Solution: Let $f(z) = \frac{coth\, z}{z-i} = \frac{cosh\, z}{(z-i)\, sinh\, z}$

The poles of $f(z)$ are given by $(z-i)\, sinh\, z = 0$

$i.e., z = i, \pm n\pi i, n$ being zero or an integer. Thus out of the many poles, $z = i$ and $z = 0$ are the only poles lying inside the given circle $|z| = 2$. Hence it is enough if we calculate the corresponding residues.

$$[Res\, f(z)]_{z=i} = \lim_{z\to i}[(z-i)f(z)]$$

$$= \lim_{z\to i}\left[(z-i)\frac{cosh\, z}{(z-i)\, sinh\, z}\right] = \lim_{z\to i} coth\, z = coth\, i$$

$$[Res\, f(z)]_{z=0} = \frac{\phi(0)}{\Psi'(0)} \text{ where } f(z) = \frac{cosh\, z}{(z-i)\, sinh\, z} = \frac{\phi(z)}{\Psi(z)}$$

$$= \left[\frac{cosh\, z}{(z-i)\, cosh\, z + \, sinh\, z}\right]_{z=0} = -\frac{1}{i}$$

∴ By Residue theorem,

$$\int_c \frac{coth\, z}{z-i} dz = 2\pi i \times \text{sum of the residues of } f(z) \text{ at the interior poles}$$

$$= 2\pi i\left(coth\, i - \frac{1}{i}\right)$$

EX. 32. Evaluate $\int_c \frac{e^{2z}}{(z-1)(z-2)} dz$ where c is the circle $|z| = 3$.

Solution: Let $f(z) = \frac{e^{2z}}{(z-1)(z-2)}$

$z = 1$ and $z = 2$ are simple poles of $f(z)$ and both poles lie inside C.

Now we have to calculate the residues at $z = 1$ and $z = 2$.

∴ $[Res\, f(z)]_{z=1} = \lim_{z\to 1}[(z-1)f(z)]$

$$= \lim_{z\to 1}\frac{e^{2z}}{z-2} = \frac{e^2}{1-2} = -e^2$$

and $[Res\, f(z)]_{z=2} = \lim_{z\to 2}[(z-1)f(z)]$

$$= \lim_{z\to 2}\frac{e^{2z}}{z-1} = \frac{e^4}{2-1} = e^4$$

∴ By Residue theorem,

$$\int_c \frac{e^{2z}}{(z-1)(z-2)}dz$$

$$= 2\pi i \times \text{sum of the residues of } f(z) \text{ at the interior poles}$$

$$= 2\pi i(-e^2 + e^4)$$

EX. 33. Evaluate $\int_c \frac{12z-7}{(2z+3)(z-1)^2}dz$ where c is the circle $x^2 + y^2 = 4$.

Solution: Let $f(z) = \frac{12z-7}{(2z+3)(z-1)^2}$

For the given function, $z = 1$ is a pole of second order and $z = -\frac{3}{2}$ is a simple pole.

Now we have to calculate the residues at $z = 1$ and $z = -\frac{3}{2}$.

∴ $$[Res\, f(z)]_{z=1} = \frac{1}{(2-1)!}\lim_{z\to1}\frac{d}{dz}[(z-1)^2 f(z)]$$

$$= \lim_{z\to1}\frac{d}{dz}\left[\frac{12z-7}{(2z+3)}\right] = 2$$

and $$[Res\, f(z)]_{z=-\frac{3}{2}} = \lim_{z\to-\frac{3}{2}}[(2z+3)f(z)]$$

$$= \lim_{z\to-\frac{3}{2}}\frac{12z-7}{(z-1)^2} = -2$$

∴ By Residue theorem,

$$\int_c \frac{12z-7}{(2z+3)(z-1)^2}dz$$

$$= 2\pi i \times \text{sum of the residues of } f(z) \text{ at the interior poles}$$

$$= 2\pi i(2-2) = 0$$

EX. 34. Evaluate $\int_c \frac{e^z}{(z^2+\pi^2)^2}dz$ where C is the circle $|z| = 4$.

Solution: Let $f(z) = \frac{e^z}{(z^2+\pi^2)^2}$

For $f(z), z = \pm i\pi$ are double poles which lie inside c.

∴ $$[Res\, f(z)]_{z=i\pi} = \frac{1}{(2-1)!}\lim_{z\to i\pi}\frac{d}{dz}[(z-i\pi)^2 f(z)]$$

$$= \lim_{z\to i\pi} \frac{d}{dz}\left[\frac{e^z}{(z+i\pi)^2}\right] = \frac{i+\pi}{4\pi^3}$$

and $\quad [Res\, f(z)]_{z=i\pi} = \frac{1}{(2-1)!} \lim_{z\to -i\pi} \frac{d}{dz}[(z+i\pi)^2 f(z)]$

$$= \lim_{z\to -i\pi} \frac{d}{dz}\left[\frac{e^z}{(z-i\pi)^2}\right] = \frac{-i+\pi}{4\pi^3}$$

∴ By Residue theorem,

$$\int_c \frac{e^z}{(z^2+\pi^2)^2} dz = 2\pi i \times \text{sum of the residues of } f(z) \text{ at the interior poles}$$

$$= 2\pi i\left(\frac{i+\pi}{4\pi^3} + \frac{-i+\pi}{4\pi^3}\right) = \frac{i}{\pi}$$

EX. 35. Evaluate $\int_c \frac{sin\, z}{z\, cos\, z} dz$ where c is the circle $|z| = \pi$ by Residue theorem.

Solution: Let $f(z) = \frac{sin\, z}{z\, cos\, z}$

The poles of $f(z)$ are given by $z\, cos\, z = 0$

$i.e.,\quad z = 0, (2n+1)\frac{\pi}{2}, n$ being zero or an integer.

$i.e.,\quad z = 0, \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots$

Of the many simple poles, $z = 0,\ z = \frac{\pi}{2}$ and $z = -\frac{\pi}{2}$ are the only poles lying inside the circles $|z| = \pi$. Hence it is enough if we calculate the corresponding residues.

Now let $\quad f(z) = \frac{sin\, z}{z\, cos\, z} = \frac{\phi(z)}{\Psi(z)}$. Then $\Psi'(z) = cos\, z - z\, sin\, z$

∴ $\quad [Res\, f(z)]_{z=0} = \frac{\phi(0)}{\Psi'(0)} = \frac{0}{1} = 0$

$$[Res\, f(z)]_{z=\frac{\pi}{2}} = \frac{\phi\left(\frac{\pi}{2}\right)}{\Psi'\left(\frac{\pi}{2}\right)} = \frac{1}{-\frac{\pi}{2}} = -\frac{2}{\pi}$$

$$[Res\, f(z)]_{z=-\frac{\pi}{2}} = \frac{\phi\left(-\frac{\pi}{2}\right)}{\Psi'\left(-\frac{\pi}{2}\right)} = \frac{-1}{-\frac{\pi}{2}} = \frac{2}{\pi}$$

∴ By Residue theorem,

$$\int_c \frac{\sin z}{z\cos z}dz = 2\pi i \times \text{sum of the residues of } f(z) \text{ at the interior poles}$$

$$= 2\pi i\left(0 - \frac{2}{\pi} + \frac{2}{\pi}\right) = 0$$

EX. 36. Evaluate $\int_c \frac{z}{(z-1)(z-2)^2}dz$ where c is the circle

$|z-2| = \frac{1}{2}$ by Residue theorem.

Solution: Let $f(z) = \frac{z}{(z-1)(z-2)^2}$

The poles of $f(z)$ are given by $(z-1)(z-2)^2 = 0, i.e., z = 1$ and $z = 2$

$z = 1$ is a simple pole and $z = 2$ is a pole of the second order.

Of these, only $z = 2$ lie within the circle $|z-2| = \frac{1}{2}$.

Now we have to calculate the residue at $z = 2$.

$$[Res\ f(z)]_{z=2} = \frac{1}{(2-1)!}\lim_{z\to 2}\frac{d}{dz}[(z-2)^2 f(z)]$$

$$= \lim_{z\to 2}\frac{d}{dz}\left(\frac{z}{z-1}\right) = \lim_{z\to 2}\frac{d}{dz}\left(1 + \frac{1}{z-1}\right) = -1$$

∴ By Residue theorem,

$$\int_c \frac{z}{(z-1)(z-2)^2}dz$$

$$= 2\pi i \times \text{sum of the residues of } f(z) \text{ at the interior poles}$$

$$= 2\pi i(-1) = -2\pi i$$

2.18. Evaluation of Definite Integrals:

One of the important applications of the theory of residues consists in the evaluation of certain types of real definite integrals. These integrals often arise in physical problems. It must be observed that a definite integral that can be evaluated by the use of Cauchy's residue theorem may be evaluated by other methods although not to easily. However, there are some simple integrals like $\int_0^\infty e^{-x^2}dx$ which cannot be evaluated by Cauchy's method. We shall now consider some integrals which can be evaluated by applying the residue theorem. These integral are evaluated by making the path of integration a suitable contour in the complex plane. This process is called *contour integration*.

2.19. Integration round the unit circle:

An integral of the type $\int_0^{2\pi} f(\cos\theta, \sin\theta)\, d\theta$, where the integrand is a rational function of $\cos\theta$ and $\sin\theta$ can be evaluated by putting $e^{i\theta} = z$.

Then $\cos\theta = \dfrac{e^{i\theta} + e^{-i\theta}}{2} = \dfrac{z + \frac{1}{z}}{2}$; $\sin\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2i} = \dfrac{z - \frac{1}{z}}{2i}$

Also $e^{i\theta} i\, d\theta = dz$ and so $d\theta = \dfrac{dz}{iz}$

Hence $\displaystyle\int_0^{2\pi} f(\cos\theta, \sin\theta)\, d\theta = \int_C F(z)\, dz$

where $F(z)$ is a rational function of z and C is the unit circle $|z| = 1$.

$$\text{But} \int_C F(z)\, dz = 2\pi i \sum R$$

where $\sum R$ denotes the sum of the residues of $F(z)$ at its poles inside C.

EX.37. By integrating round a circle of unit radius, show that

$$\int_0^{2\pi} \frac{\cos 3\theta}{5 - 4\cos\theta}\, d\theta = \frac{\pi}{12}.$$

Solution: Let $z = e^{i\theta}$. Then $dz = e^{i\theta} i\, d\theta = iz\, d\theta$.

Therefore $d\theta = \dfrac{dz}{iz}$ and $\cos\theta = \dfrac{z + \frac{1}{z}}{2}$

$$\cos 3\theta = \frac{e^{i.3\theta} + e^{-i.3\theta}}{2} = \frac{z^3 + \frac{1}{z^3}}{2}$$

Let C be the unit circle $|z| = 1$.

The given integral I is

$$= \int_C \frac{\left(z^3 + \frac{1}{z^3}\right)}{2\left[5 - \frac{4\left(z + \frac{1}{z}\right)}{2}\right]} \frac{dz}{iz}$$

$$= \int_C \frac{z^6+1}{2\,z^3\left[5-2z-\frac{2}{z}\right]}\frac{dz}{iz}$$

$$= \frac{1}{2i}\int_C \frac{z^6+1}{z^3(5z-2z^2-2)}\;dz$$

$$= -\frac{1}{2i}\int_C \frac{z^6+1}{z^3(2z^2-5z+2)}\;dz$$

$$= -\frac{1}{2i}\int_C \frac{z^6+1}{z^3(2z-1)(z-2)}\;dz$$

$$= -\frac{1}{2i}\int_C f\,(z)\,dz \qquad (1)$$

The poles of $f(z)$ are:

(i) $z=0$, a pole of order 3 (ii) $z=\frac{1}{2}$ and (iii) $z=2$.

Of these only $z=0$ and $z=\frac{1}{2}$ are inside the unit circle. Hence we have to calculate the corresponding residues.

Residue of $f(z)$ at $z=\frac{1}{2}$ is

$$= \lim_{z\to\frac{1}{2}} \frac{\left(z-\frac{1}{2}\right)(z^6+1)}{z^3(2z-1)(z-2)} = \lim_{z\to\frac{1}{2}} \frac{(z^6+1)}{2z^3(z-2)} = -\frac{65}{24}$$

It will be easier to get the residue at the multiple pole $z=0$, by expansion of $f(z)$.

$$f(z) = \frac{z^6+1}{z^3(2z-1)(z-2)} = \frac{\left(z^3+\frac{1}{z^3}\right)}{(1-2z)(2-z)}$$

$$= \frac{\left(z^3+\frac{1}{z^3}\right)}{2(1-2z)\left(1-\frac{z}{2}\right)}$$

$$= \frac{1}{2}\left(z^3+\frac{1}{z^3}\right)(1-2z)^{-1}\left(1-\frac{z}{2}\right)^{-1}$$

$$= \frac{1}{2}\left(z^3+\frac{1}{z^3}\right)(1+2z+4z^2+\cdots)\left(1+\frac{z}{2}+\frac{z^2}{4}+\cdots\right)$$

$$= \frac{1}{2}\left(z^3+\frac{1}{z^3}\right)(1+2z+4z^2+\cdots)\left(1+\frac{z}{2}+\frac{z^2}{4}+\cdots\right)$$

$$= \frac{1}{2}\left(z^3 + \frac{1}{z^3}\right)\left[1 + \frac{5z}{2} + z^2\left(\frac{1}{4} + 1 + 4\right) + \cdots\right]$$

Coefficient of $\frac{1}{z}$ in the right hand side product

$$= \frac{1}{2} \times 1 \times \left(\frac{1}{4} + 1 + 4\right) = \frac{21}{8}$$

Hence by the residue theorem,

$\int_C f(z)\ dz = 2\pi i \times$ sum of the above two residues

$$= 2\pi i\left(-\frac{65}{24} + \frac{21}{8}\right) = -\frac{\pi i}{6}$$

Hence substituting in (1),

$$I = -\frac{1}{2i} \times -\frac{\pi i}{6} = \frac{\pi}{12}$$

EX.38. Evaluate by contour integration

$$\int_0^{2\pi} \frac{sin^2\theta}{a + b\ cos\ \theta} d\theta, a > b > 0.$$

Solution: Let $z = e^{i\theta}$ and C be the unit circle $|z| = 1$.

Then $dz = e^{i\theta} i\ d\theta = iz\ d\theta$ or $d\theta = \frac{dz}{iz}$

$$cos\ \theta = \frac{z + \frac{1}{z}}{2} \text{ and } sin\ \theta = \frac{z - \frac{1}{z}}{2i}$$

The given integral I is

$$= \int_C \frac{\left(z - \frac{1}{z}\right)^2}{4i^2\left[a + \frac{b\left(z + \frac{1}{z}\right)}{2}\right]} \frac{dz}{iz}$$

$$= \int_C \frac{(z^2 - 1)^2 . 2}{-4z^2\left(2a + bz + \frac{b}{z}\right)} \frac{dz}{iz}$$

$$= -\frac{1}{2i} \int_C \frac{(z^2 - 1)^2}{z^2(bz^2 + 2az + b)} dz$$

$$= -\frac{1}{2i} \int_C \frac{(z^2 - 1)^2}{z^2 . b(z - p)(z - q)} dz$$

where p and q are the roots of $bz^2 + 2az + b = 0$.

$$= -\frac{1}{2i}\int_C f(z)\,dz \qquad (1)$$

The poles of $f(z)$ are:

(i) $z = 0$, a double pole (ii) $z = p$ and $z = q$.

Solving $bz^2 + 2az + b = 0$, we have

$$z = \frac{-2a \pm \sqrt{4a^2 - 4b^2}}{2b}$$

$$= \frac{-a \pm \sqrt{a^2 - b^2}}{b}$$

$$= \frac{-a + \sqrt{a^2 - b^2}}{b}, \frac{-a - \sqrt{a^2 - b^2}}{b}$$

$$= -\frac{a}{b} + \frac{\sqrt{a^2 - b^2}}{b}, -\frac{a}{b} - \frac{\sqrt{a^2 - b^2}}{b}$$

As $a > b, \frac{a}{b} > 1$ and the root $-\frac{a}{b} - \frac{\sqrt{a^2-b^2}}{b}$ is numerically greater than 1.

Hence the pole $z = -\frac{a}{b} + \frac{\sqrt{a^2-b^2}}{b}$ alone is inside the unit circle. We can take this as p, and the other pole $z = -\frac{a}{b} + \frac{\sqrt{a^2-b^2}}{b}$ as q.

Also $pq =$ product of the two values of $z = 1$.

We have now to calculate the residues of $f(z)$at $z = p$ and $z = 0$.

Residue at $z = p$is

$$= \lim_{z\to p} \frac{(z-p)(z^2-1)^2}{z^2.\,b(z-p)(z-q)}$$

$$= \lim_{z\to p} \frac{(z^2-1)^2}{z^2.\,b(z-q)}$$

$$= \frac{(p^2-1)^2}{p^2.\,b(p-q)} = \frac{(p^2-pq)^2}{b\,p^2(p-q)} = \frac{p^2(p-q)^2}{b\,p^2(p-q)}$$

$$= \frac{p-q}{b} = \frac{1}{b}\left[\frac{-a+\sqrt{a^2-b^2}}{b} - \frac{-a-\sqrt{a^2-b^2}}{b}\right]$$

$$= \frac{1}{b}\left[\frac{-a+\sqrt{a^2-b^2}+a+\sqrt{a^2-b^2}}{b}\right]$$

$$= \frac{2\sqrt{a^2-b^2}}{b^2}$$

Residue at $z = 0$is

$$= \lim_{z\to 0} \frac{d}{dz} \frac{z^2(z^2-1)^2}{z^2(bz^2+2az+b)}$$

$$= \lim_{z\to 0} \frac{(bz^2+2az+b)2(z^2-1)2z-(z^2-1)^2(2bz+2a)}{(bz^2+2az+b)^2}$$

$$= -\frac{2a}{b^2}$$

Hence by residue theorem,

$$\int_C f(z)\,dz = 2\pi i \times \text{sum of the above two residues}$$

$$= 2\pi i \times \left(\frac{2\sqrt{a^2-b^2}}{b^2} - \frac{2a}{b^2}\right) = \frac{4\pi i}{b^2}\left[\sqrt{a^2-b^2}-a\right]$$

Hence substituting in (1),

$$I = -\frac{1}{2i} \times \frac{4\pi i}{b^2}\left[\sqrt{a^2-b^2}-a\right]$$

$$= \frac{2\pi}{b^2}\left[a-\sqrt{a^2-b^2}\right]$$

EX.39. Show by the method of residues, $\int_0^{\pi} \frac{d\theta}{a+b\cos\theta} = \frac{\pi}{\sqrt{a^2-b^2}}$ $(a > b > 0)$.

or Show that $\int_0^{2\pi} \frac{d\theta}{a+b\cos\theta} = \frac{2\pi}{\sqrt{a^2-b^2}}$ $(a > b > 0)$.

Solution: We have $\int_0^{\pi} \frac{d\theta}{a+b\cos\theta} = \frac{1}{2}\int_0^{2\pi} \frac{d\theta}{a+b\cos\theta}$ (1)

Let c be the circle $|z| = 1$.

Put $z = e^{i\theta}$, so that $d\theta = \frac{dz}{iz}$ and $\cos\theta = \frac{z+\frac{1}{z}}{2} = \frac{z^2+1}{2z}$.

$$\therefore \quad \int_0^{2\pi} \frac{d\theta}{a+b\cos\theta} = \int_c \frac{1}{a+b\left[\frac{z^2+1}{2z}\right]} \cdot \frac{dz}{iz}$$

$$= \frac{1}{i}\int_c \frac{2}{bz^2+2az+b}\,dz$$

$$= \frac{2}{i}\int_c \frac{1}{bz^2+2az+b}\,dz$$

$$= \frac{2}{i}\int_c f(z)dz$$

Now, the poles of $f(z)$ are the roots of $bz^2 + 2az + b = 0$, so

$$z = \frac{-2a \pm \sqrt{4a^2 - 4b^2}}{2b} = \frac{-a \pm \sqrt{a^2 - b^2}}{b}$$

are the poles.

$$\alpha = \frac{-a + \sqrt{a^2 - b^2}}{b}, \beta = \frac{-a - \sqrt{a^2 - b^2}}{b}$$

Since $a > b > 0$, we have $|\beta| > 1$. But the product of the roots is 1.

$i.e., |\alpha\beta| > 1$ so that $|\alpha| < 1$.

Thus, $z = \alpha$ is the only simple pole lies inside c and so

$$f(z) = \frac{1}{b(z-\alpha)(z-\beta)}$$

$$\therefore \qquad [Res\, f(z)]_{z=\alpha} = \lim_{z\to\alpha}(z-\alpha)\, f(z)$$

$$= \lim_{z\to\alpha}(z-\alpha)\frac{2}{i}\frac{1}{b(z-\alpha)(z-\beta)}$$

$$= \frac{2}{i}\lim_{z\to\alpha}\frac{1}{b(z-\beta)} == \frac{2}{i}\frac{1}{b(z-\alpha)}$$

$$= \frac{2}{i}\frac{1}{b\left[\frac{2\sqrt{a^2-b^2}}{b}\right]} = \frac{1}{i\sqrt{a^2-b^2}}$$

Thus, $$\int_0^{2\pi} \frac{d\theta}{a + b\cos\theta} = 2\pi i\left(\frac{1}{i\sqrt{a^2-b^2}}\right) = \frac{2\pi}{\sqrt{a^2-b^2}}$$

Hence $$\int_0^{\pi} \frac{d\theta}{a + b\cos\theta} = \frac{\pi}{\sqrt{a^2-b^2}} \quad \text{[From (1)]}$$

Note: **1**. In place of a and b whatever the values we take with $a > b > 0$, we can solve the integrals.

2. Similarly we can prove

$$\int_0^{2\pi} \frac{d\theta}{a + b\sin\theta} = \frac{2\pi}{\sqrt{a^2-b^2}}$$

3. Taking $a = 3$ and $b = 2$ in the above example, we get $\int_0^{\pi} \frac{d\theta}{3 + 2\cos\theta} = \frac{\pi}{\sqrt{5}}$.

4. Observe that $\int_0^{\pi} \frac{d\theta}{a + b\sin\theta} = \frac{\pi}{\sqrt{a^2-b^2}}$

EX. 40. Evaluate by contour integration $\int_0^{2\pi} \frac{d\theta}{2 - \sin\theta}$.

Solution: Write $z = re^{i\theta}$, then $d\theta = \frac{dz}{iz}$ and so $\sin\theta = \frac{z - \frac{1}{z}}{2i} = \frac{z^2 - 1}{2iz}$.

$$\therefore \quad \int_0^{2\pi} \frac{d\theta}{2 - \sin\theta} = \int_c \frac{1}{2 - \left(\frac{z^2-1}{2iz}\right)} \frac{dz}{iz}, \text{where } c: |z| = 1$$

$$= \int_c \frac{2iz}{4iz - z^2 + 1} \frac{dz}{iz}$$

$$= -2 \int_c \frac{1}{z^2 - 4iz - 1}\, dz$$

$$= -2 \int_c f(z)\, dz, \text{where } f(z) = \frac{1}{z^2 - 4iz - 1}$$

Now, the poles of $f(z)$ are the roots of $z^2 - 4iz - 1 = 0$.

The roots are $z = \frac{4i \pm \sqrt{(4i)^2 + 4}}{2} = \frac{4i \pm \sqrt{-16 + 4}}{2} = (2 \pm \sqrt{3})i$

Denote $\alpha = (2 + \sqrt{3})i$ and $\beta = (2 - \sqrt{3})i$ and α and β are the simple poles.

Observe that $\alpha\beta = -1$ and $|\alpha| > 1$ so $|\beta| < 1$.

Therefore, β is only pole lie inside unit circle c.

Now, $\quad [Res\, f(z)]_{z=\beta} = \frac{1}{\beta - \alpha} = -\frac{1}{2\sqrt{3}i}$

$\therefore$ By Residue theorem, we have

$$\int_c f(z)\, dz = 2\pi i\left(-\frac{1}{2\sqrt{3}i}\right) = -\frac{\pi}{\sqrt{3}}$$

Hence, $\quad \int_0^{2\pi} \frac{d\theta}{2 - \sin\theta} = -2\left(-\frac{\pi}{\sqrt{3}}\right) = \frac{2\pi}{\sqrt{3}}$

EX. 41. Show that $\int_0^{\pi} \frac{d\theta}{a^2 + \sin^2\theta} = \frac{\pi}{a\sqrt{a^2 + 1}}$ for $a > 0$.

Solution: Write $I = \int_0^{\pi} \frac{d\theta}{a^2 + \sin^2\theta} = \int_0^{\pi} \frac{2d\theta}{2a^2 + (1 - \cos 2\theta)}$

Put $2\theta = \phi$; $2\, d\theta = d\phi$

$\therefore \qquad I = \int_0^{2\pi} \frac{d\phi}{2a^2 + 1 - \cos\phi}$

Put $\quad z = e^{i\phi}$; $d\phi = \frac{dz}{iz}$

$\therefore \qquad I = \int_c \frac{1}{2a^2 + 1 - \left(\frac{z^2+1}{2z}\right)} \cdot \frac{dz}{iz} \quad$ where $\; c: |z| = 1$

$$= \frac{1}{i} \int_c \frac{2z}{(4a^2+2)z - z^2 - 1} \cdot \frac{dz}{z}$$

$$= -\frac{2}{i} \int_c f(z)\; dz$$

Now, the poles of $f(z)$ are given by

$$z = \frac{(4a^2+2) \pm \sqrt{[-(4a^2+2)]^2 - 4}}{2} = (2a^2+1) \pm 2a\sqrt{a^2+1}$$

Let $\alpha = (2a^2+1) + 2a\sqrt{a^2+1}$ and $\beta = (2a^2+1) - 2a\sqrt{a^2+1}$

Observe that $|\alpha| > 1$ and since $\alpha\beta = 1$, we have $|\beta| < 1$.

So β is the only pole lies within c.

$\therefore \qquad [Res\, f(z)]_{z=\beta} = \lim_{z\to\beta} (z-\beta)\; f(z)$

$$= \lim_{z\to\beta} \frac{1}{z-\alpha} = \frac{1}{\beta - \alpha} = \frac{1}{-4a\sqrt{a^2+1}}$$

$\therefore \qquad$ By Residue theorem, we have

$$I = -\frac{2}{i} \cdot 2\pi i \cdot \frac{1}{-4a\sqrt{a^2+1}} = \frac{\pi}{a\sqrt{a^2+1}}$$

Note: $\int_0^{\pi} \frac{d\theta}{a^2 + \cos^2\theta} = \frac{\pi}{a\sqrt{a^2+1}}$

EX. 42. Show that $\int_0^{2\pi} \frac{d\theta}{2 + \cos\theta} = \frac{2\pi}{\sqrt{3}}$.

Solution: Let $I = \int_0^{2\pi} \frac{d\theta}{2 + \cos\theta}$

On the unit circle $|z| = 1$, we have

$$z = e^{i\theta}, d\theta = \frac{dz}{iz} \text{ } and \text{ } cos\, \theta = \frac{1}{2}\left(z + \frac{1}{z}\right) = \frac{z^2+1}{2z}$$

Substituting these values, we get

$$I = \int_c \frac{1}{2 + \frac{z^2+1}{2z}} \cdot \frac{dz}{iz} = \frac{2}{i}\int_c \frac{dz}{z^2+4z+1}$$

Where c is the unit circle $|z| = 1$.

The integrand $f(z) = \frac{1}{z^2+4z+1}$ has simple poles given by

$$z = \frac{-4 \pm \sqrt{16-4}}{2} = -2 \pm \sqrt{3}$$

Of these only $z = -2 + \sqrt{3}$ lies inside c.

∴ $$[Res\, f(z)]_{z=-2+\sqrt{3}} = \lim_{z\to -2+\sqrt{3}} \left\{z - \left(-2+\sqrt{3}\right)\right\} f(z)$$

$$= \lim_{z\to -2+\sqrt{3}} \left[\frac{1}{z+2+\sqrt{3}}\right] = \frac{1}{2\sqrt{3}}$$

∴ By Residue theorem, we have

$$I = \frac{2}{i} \text{ . } 2\pi i \times (\text{Sum of the residues of } f(z) \text{ at the poles within c})$$

$$= 2\pi i \cdot \frac{2}{i} \cdot \frac{1}{2\sqrt{3}} = \frac{2\pi}{\sqrt{3}}$$

EX.43. Use the method of contour integration to prove that

$$\int_0^{2\pi} \frac{d\theta}{1 + a^2 + 2a\cos\theta} = \frac{2a\pi}{1-a^2}, 0 < a < 1$$

Solution: Let $z = e^{i\theta}$ and C be the unit circle $|z| = 1$.

Then $dz = e^{i\theta} i\, d\theta$ or $d\theta = \frac{dz}{iz}$ and $cos\, \theta = \frac{1}{2}\left(z + \frac{1}{z}\right) = \frac{z^2+1}{2z}$

The given integral I is

$$= \int_C \frac{1}{1 + a^2 - 2a\,\frac{z^2+1}{2z}} \cdot \frac{dz}{iz} = \frac{1}{i}\int_C \frac{dz}{az^2 - (1+a^2)z + a}$$

The poles of the integrand are given by

$$\frac{(1+a^2) \pm \sqrt{(1-a^2)^2}}{2a} \text{ } i.e., \frac{1}{a} \text{ and } a.$$

Of these poles only $z_0 = a$ lies inside C ($\because$ $a < 1$)

Residue at $z = a$ is

$$= \lim_{z\to a}(z-a)\frac{1}{\left(z-\frac{1}{a}\right)(z-a)}$$

$$= \lim_{z\to a}\frac{1}{\left(z-\frac{1}{a}\right)} = \frac{1}{a-\frac{1}{a}} = \frac{a}{a^2-1}$$

By Residue theorem, $\quad I = -\frac{1}{i}\,.2\pi i\,.\frac{a}{a^2-1} = \frac{2a\pi}{1-a^2}$

EX. 44. Evaluate $\int_0^{2\pi}\frac{d\theta}{(5-3\sin\theta)^2}$ using residue theorem.

Solution: Let $z = e^{i\theta}$ and C be the unit circle $|z| = 1$.

Then $dz = e^{i\theta} i\, d\theta$ or $d\theta = \frac{dz}{iz}$ and $\sin\theta = \frac{1}{2i}\left(z-\frac{1}{z}\right) = \frac{z^2-1}{2iz}$

Substituting these values, we get

$$I = \int_0^{2\pi}\frac{d\theta}{(5-3\sin\theta)^2} = \int_C \frac{dz}{iz\left[5-3\left(\frac{z^2-1}{2iz}\right)\right]^2}$$

$$= \int_C \frac{dz}{iz[10iz-3(z^2-1)]^2}\,.4i^2z^2$$

$$= 4i\int_C \frac{z\,dz}{(3z^2-10iz-3)^2}$$

$$= 4i\int_C f(z)\,dz, \text{where } f(z) = \frac{z}{(3z^2-10iz-3)^2} \qquad (1)$$

Now we have to evaluate $\int_C f(z)\,dz$

The poles of $f(z)$ are given by $\quad 3z^2 - 10iz - 3 = 0$

$i.e.,\quad z = \frac{10i \pm \sqrt{-100+36}}{6} = \frac{10i \pm 8i}{6}$

$i.e.,\quad z = \frac{10i+8i}{6} = 3i$ and $z = \frac{10i-8i}{6} = \frac{i}{3}$

Out of these two poles only the pole $z = \frac{i}{3}$ of order 2 lies within the circle $|z| = 1$.

Residue of $f(z)$ at $z = \frac{i}{3}$ is $= \frac{1}{(2-1)!}\lim_{z\to\frac{i}{3}}\frac{d}{dz}\left[\left(z-\frac{i}{3}\right)^2 f(z)\right]$

$$= \lim_{z\to\frac{i}{3}}\frac{d}{dz}\left[\frac{1}{9}(3z-i)^2\frac{z}{(3z^2-10iz-3)^2}\right]$$

$$= \lim_{z \to \frac{i}{3}} \frac{d}{dz}\left[\frac{1}{9}(3z-i)^2 \frac{z}{(3z-i)^2(z-3i)^2}\right]$$

$$= \lim_{z \to \frac{i}{3}} \frac{d}{dz}\left[\frac{1}{9}\frac{z}{(z-3i)^2}\right] = -\frac{5}{256}$$

Thus by Residue theorem,

$$\int_C f(z)\,dz = 2\pi i \times (\text{Sum of the residues})$$

$$= 2\pi i\left(-\frac{5}{256}\right) = -\frac{10\pi i}{256} \qquad (2)$$

Hence $\quad I = 4i\left(-\frac{10\pi i}{256}\right) \quad$ [From (1) and (2)]

$$\therefore \quad \int_0^{2\pi} \frac{d\theta}{(5-3\sin\theta)^2} = \frac{5\pi}{32}$$

EX. 45. Show that $\displaystyle\int_0^{2\pi} \frac{d\theta}{a+b\sin\theta} = \int_0^{2\pi} \frac{d\theta}{a+b\cos\theta} = \frac{2\pi}{\sqrt{a^2-b^2}}, a > b > 0$

using residue theorem.

Solution: Let $\quad I = \displaystyle\int_0^{2\pi} \frac{d\theta}{a+b\sin\theta}$

Let $z = e^{i\theta}$ and C be the unit circle $|z| = 1$.

Then $dz = e^{i\theta} i\,d\theta$ or $d\theta = \dfrac{dz}{iz}$ and $\sin\theta = \dfrac{1}{2i}\left(z - \dfrac{1}{z}\right) = \dfrac{z^2-1}{2iz}$

$$\therefore \qquad I = \int_C \frac{1}{a + \frac{b}{2i}\left(z - \frac{1}{z}\right)} \cdot \frac{dz}{iz} = \int_C \frac{2zi}{2iaz + b(z^2-1)} \cdot \frac{dz}{iz}$$

$$= 2\int_C \frac{dz}{bz^2 + 2iaz - b} = \frac{2}{b}\int_C \frac{dz}{z^2 + \frac{2a}{b}iz - 1}$$

$$= \frac{2}{b}\int_C f(z)\,dz, \text{where } f(z) = z^2 + \frac{2a}{b}iz - 1 = \frac{1}{(z-\alpha)(z-\beta)}$$

where $\alpha + \beta = -\dfrac{2ai}{b}$ and $\alpha\beta = -1$

The poles of $f(z)$ are given by $z^2 + \dfrac{2a}{b}iz - 1 = 0$

$$\therefore \quad z = \frac{1}{2}\left[-\frac{2ai}{b} \pm \sqrt{\frac{-4a^2}{b^2}+4}\right] = \frac{1}{2}\left[-\frac{2ai}{b} \pm \frac{2i}{b}\sqrt{a^2-b^2}\right]$$

$$= \frac{i}{b}\left[-a \pm \sqrt{a^2-b^2}\right]$$

Let the poles of $f(z)$ be

$$\alpha = \frac{i}{b}\left(-a + \sqrt{a^2-b^2}\right) \text{ and } \beta = \frac{i}{b}\left(-a - \sqrt{a^2-b^2}\right)$$

Out of these poles $z = \alpha$ lies within the circle $|z| = 1$.

Residue of $f(z)$ at $z = \alpha$ is

$$[Res\, f(z)]_{z=\alpha} = \lim_{z\to\alpha}[(z-\alpha)f(z)]$$

$$= \lim_{z\to\alpha}\left[(z-\alpha)\frac{1}{(z-\alpha)(z-\beta)}\right] = \frac{1}{\alpha-\beta}$$

$$= \frac{1}{\sqrt{(\alpha+\beta)^2-4\alpha\beta}} = \frac{1}{\sqrt{\frac{-4a^2}{b^2}+4}}$$

$$= \frac{b}{\sqrt{-4a^2+4b^2}} = \frac{b}{2i\sqrt{a^2-b^2}}$$

Thus by Residue theorem,

$$\int_C f(z)\,dz = 2\pi i \,. \frac{b}{2i\sqrt{a^2-b^2}} = \frac{\pi b}{\sqrt{a^2-b^2}}$$

Hence $\quad I = \frac{2}{b}\int_C f(z)\,dz = \frac{2}{b}\,.\frac{\pi b}{\sqrt{a^2-b^2}}$

$$\therefore \quad \int_0^{2\pi} \frac{d\theta}{a+b\sin\theta} = \frac{2\pi}{\sqrt{a^2-b^2}}$$

Similarly we can prove that $\int_0^{2\pi} \frac{d\theta}{a+b\cos\theta} = \frac{2\pi}{\sqrt{a^2-b^2}}$

$$\therefore \quad \int_0^{2\pi} \frac{d\theta}{a+b\sin\theta} = \int_0^{2\pi} \frac{d\theta}{a+b\cos\theta} = \frac{2\pi}{\sqrt{a^2-b^2}}$$

2.20. Evaluation of certain real integrals between the limits $-\infty$ *and* ∞:

We shall now prove the following theorem:

Let $Q(z)$ be a function satisfying the following conditions:

(a) $Q(z)$ is analytic in the upper half of the z-plane, except at a finite number of poles.

(b) $Q(z)$has no poles on the real axis.

(c) Let $|z\,Q(z)| = 0$ as $|z| \to \infty$ through the values of z such that $0 \le amp\, z \le \pi$.

Then $\int_{-\infty}^{\infty} Q(x)\,dx = 2\pi i \times$ sum of the residues of$Q(z)$at its poles which lie on theupper half plane.

Consider a semicircle with centre at $z = 0$ and with radius R sufficiently large so as to include all the poles of $Q(z)$ which lie in the upper half plane as shown in the figure.

Then by Cauchy's residue theorem, we have

$$\int_{C_1+C_2} Q(z)\,dz = 2\pi i \times \sum \text{residues of } Q(z) \text{at all poles within} C_1 + C_2$$

$$\int_{-R}^{R} Q(z)\,dz + \int_{C_2} Q(z)\,dz = 2\pi i \times \sum \text{residues} \qquad (1)$$

Now , in $\int_{C_2} Q(z)\,dz, put\ z = R\,e^{i\theta}$

Then $dz = R\,e^{i\theta}\,i\,d\theta = iz\,d\theta$

Therefore

$$\left|\int_{C_2} Q(z)\,dz\right| = \int_0^{\pi} |Q(z)\,iz|\,d\theta = \int_0^{\pi} |Q(z)\,z|\,d\theta \qquad (2)$$

Now, by condition (c) of this theorem,

Let $|z\,Q(z)| = 0$ as $|z| i.e., R \to \infty$.

Hence if R is large enough, we can find an arbitrary small positive quantity δ such that $|z\,Q(z)| < \delta$.

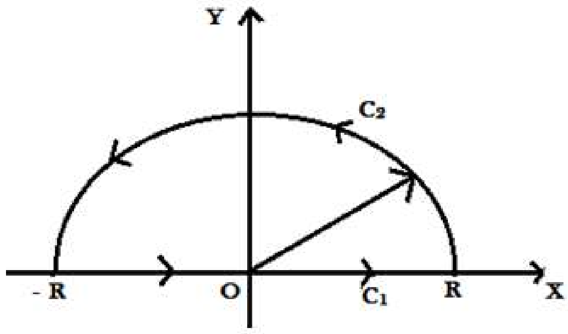


Figure 2

Then, from (2), we have the inequality

$$\left|\int_{C_2} Q(z)\,dz\right| < \int_0^{\pi} \delta\,d\theta$$

$i.e., < \delta\pi, i.e., <$ a very small quantity

This means that $\int_{C_2} Q(z)\,dz = 0 \; as \; R \to \infty.$

Hence taking limits in (1), as $R \to \infty$, we have

$$\int_{-\infty}^{\infty} Q(x)\,dx = 2\pi i \sum \text{residues}$$

Note 1: In problems it will be easier to directly show that $\int_{C_2} Q(z)\,dz \to 0$ as $z \to \infty$

instead of applying the theorem.

Note 2: The theorem will be specially useful in the case when $Q(z)$is a rational function.

Note 3: In particular, if $Q(z) = \frac{p(z)}{q(z)}$ such that the degree of the denominator is greater than that of the numerator by at least two and $q(z)$ has no poles on the real axis, the conditions of the theorem are automatically satisfied. Hence we have the following important result:

If $p(z)$ and $q(z)$ are real polynomials such that the degree of $q(z)$ is greater than that of $p(z)$ by at least two and if $q(z) = 0$ has no real roots, then

$$\int_{-\infty}^{\infty} \frac{p(z)}{q(z)}\,dz = 2\pi i \times \text{sum of the residues of} \frac{p(z)}{q(z)} \text{at its poles in the upper half of the}$$

$z -$ plane.

EX. 46. Evaluate by contour integration $\int_{0}^{\infty} \frac{dz}{x^4 + a^4}$.

Solution: *Here we consider* $\int_C \frac{dz}{z^4 + a^4} = \int_C Q(z)\, dz$

where C is the contour consisting of the semicircle C_2 of radius R and C_1 is the segment of the real axis from $-R$ to R as shown in the above figure.

$$\text{Then} \int_{-R}^{R} Q(x)\ dx + \int Q(z)\ dz = 2\pi i \sum \text{residues of } Q(z) \text{in the upper half plane} \qquad (1)$$

$$\text{Now } |z| = R \text{ on} C_2$$

$$\text{And } |z^4 + a^4| \geq |z|^4 - a^4$$

$$i.e., \geq R^4 - a^4$$

$$\text{Therefore } \left|\frac{1}{z^4 + a^4}\right| \leq \frac{1}{R^4 - a^4}$$

$$\text{Hence } \left|\int_{C_2} Q(z) dz\right| = \left|\int_{C_2} \frac{1}{z^4 + a^4} dz\right|$$

$$\leq \int_{C_2} \frac{1}{R^4 - a^4} |dz|$$

$$\leq \frac{\pi . R}{R^4 - a^4}$$

and this approaches zero as $R \to \infty$.

$$\text{Therefore } \int_{C_2} Q(z) dz = 0 \text{ in the limit as } R \to \infty$$

Hence taking limit in (1), as $R \to \infty$, we have

$$\int_{-\infty}^{\infty} Q(z) dz = 2\pi i \sum residues\ of\ Q(z) (2)$$

To get the residues of $Q(z)$, we solve the equation $z^4 + a^4 = 0$.

$i.e., z^4 = -a^4 = a^4 e^{i\pi}, a^4 e^{i3\pi}, a^4 e^{i5\pi}, a^4 e^{i7\pi}$

Therefore $z = ae^{\frac{1}{4}i\pi}, ae^{\frac{1}{4}i3\pi}, ae^{\frac{1}{4}i5\pi}, ae^{\frac{1}{4}i7\pi}$

Of these four poles, the arguments of the first two points only lie between 0 and π. Hence the first two alone are in the upper half plane.

Residue of $Q(z) = \dfrac{1}{z^4 + a^4}$ at $z = ae^{\frac{1}{4}i\pi}$ is

$$= \lim_{z \to ae^{\frac{1}{4}i\pi}} \frac{1}{4z^3}$$

$$= \lim_{z \to ae^{\frac{1}{4}i\pi}} \frac{z}{4z^4} = \frac{ae^{\frac{1}{4}i\pi}}{-4a^4} = -\frac{e^{\frac{1}{4}i\pi}}{4a^3}$$

Similarly Residue of $Q(z) = \dfrac{1}{z^4 + a^4}$ at $z = ae^{\frac{1}{4}i3\pi}$ is

$$= -\frac{e^{\frac{1}{4}i3\pi}}{4a^3} = -\frac{e^{i\left(\pi - \frac{1}{4}\pi\right)}}{4a^3} = -\frac{e^{i\pi}.e^{-\frac{1}{4}i\pi}}{4a^3} = \frac{e^{-\frac{1}{4}i\pi}}{4a^3}$$

Sum of the two residues

$$= -\frac{e^{\frac{1}{4}i\pi}}{4a^3} + \frac{e^{-\frac{1}{4}i\pi}}{4a^3} = -\frac{1}{4a^3}\left(e^{\frac{1}{4}i\pi} - e^{-\frac{1}{4}i\pi}\right)$$

$$= -\frac{1}{4a^3} \times 2i\,sin\,\frac{\pi}{4} \qquad (3)$$

Hence substituting (3) in (2), we get

$$\int_{-\infty}^{\infty} Q(z)dz = 2\pi i \times -\frac{1}{4a^3} \times 2i\,sin\,\frac{\pi}{4}$$

$$= \frac{\pi}{a^3}.\frac{1}{\sqrt{2}}$$

$$\int_{-\infty}^{\infty} \frac{1}{z^4 + a^4}dz = \frac{\pi}{a^3\sqrt{2}}$$

Since $\frac{1}{z^4+a^4}$ is an even function of z, we have

$$\int_{-\infty}^{\infty} \frac{1}{z^4 + a^4}dz = 2\int_{0}^{\infty} \frac{1}{z^4 + a^4}dz = \frac{\pi}{a^3\sqrt{2}}$$

$$\text{Therefore} \int_{0}^{\infty} \frac{1}{z^4 + a^4}dz = \frac{\pi}{2\,a^3\sqrt{2}}$$

EX. 47. Evaluate $\displaystyle\int_{0}^{\infty} \frac{dx}{(x^2 + a^2)^2}$.

Solution: Since $\dfrac{1}{(x^2 + a^2)^2}$ is an even function of x, we have

$$\int_0^\infty \frac{dx}{(x^2+a^2)^2} = \frac{1}{2}\int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)^2} \qquad (1)$$

Here we consider $\int_C \frac{dz}{(z^2+a^2)^2} = \int_C f(z)dz$

where C is the contour consisting of the semi-circle C_R of radius R together with the part of the real axis from $-R$ to R.

The integrand has two poles of order 2 at $z = ia$ and $z = -ia$. But $z = ia$ only lies inside the semi-circle of the contour C.

∴ By Residue theorem,

$$\int_C f(z)dz = 2\pi i \times (\text{sum of the residues})$$

$$= 2\pi i \times [Res\, f(z)]_{z=ai}$$

$$= 2\pi i \times \lim_{z\to ai} \frac{d}{dz}\left\{(z-ai)^2 \frac{1}{(z-ai)^2(z+ai)^2}\right\}$$

$$= 2\pi i \times \lim_{z\to ai} \frac{d}{dz}\left\{\frac{1}{(z+ai)^2}\right\}$$

$$= 2\pi i \times \lim_{z\to ai} \frac{-2}{(z+ai)^3}$$

$$= 2\pi i \times \frac{-2}{(2ai)^3} = \frac{\pi}{2a^3}$$

$$i.e., \int_{-R}^{R} f(x)dx + \int_{C_R} f(z)dz = \frac{\pi}{2a^3}$$

$$i.e., \int_{-R}^{R} \frac{dx}{(x^2+a^2)^2} + \int_{C_R} \frac{dz}{(z^2+a^2)^2} = \frac{\pi}{2a^3} \qquad (2)$$

Now $\left|\int_{C_R} \frac{dz}{(z^2+a^2)^2}\right| \le \int_{C_R} \frac{|dz|}{|(z^2+a^2)^2|}$

$$\le \frac{1}{(R^2-a^2)^2}\int_0^\pi R\,d\theta \quad \left[\begin{array}{l}\because\ |z^2+a^2| > |z|^2 - |-a|^2 \text{ and } z = Re^{i\theta} \\ \Rightarrow dz = Re^{i\theta}.\,id\theta = izd\theta,\ \ |dz| = Rd\theta\end{array}\right]$$

$$= \frac{R\pi}{(R^2-a^2)^2}$$

and this $\to 0$ as $R \to \infty$

$$\therefore \int_{C_R} \frac{dz}{(z^2+a^2)^2} \to 0 \text{ as } R \to \infty$$

Hence by making $R \to \infty$, equation (2) becomes

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)^2} = \frac{\pi}{2a^3}$$

or $$\int_{0}^{\infty} \frac{dx}{(x^2+a^2)^2} = \frac{\pi}{4a^3} \text{ , from (1)}$$

Note: Evaluate $\int_{0}^{\infty} \frac{dx}{(x^2+1)^2}$ using Residue theorem.

Putting $a = 1$ in the above problem, we get $\int_{0}^{\infty} \frac{dx}{(x^2+1)^2} = \frac{\pi}{4}$.

EX. 48. Using the method of contour integration, prove that $\int_{0}^{\infty} \frac{dx}{x^6+1} = \frac{\pi}{3}$.

Solution: Since integrand is an even function, we have $\int_{-\infty}^{\infty} \frac{dx}{x^6+1} = \frac{1}{2}\int_{0}^{\infty} \frac{dx}{x^6+1}$

Consider $\int_{C} \frac{dx}{z^6+1} = \int_{C} f(z)dz$

where C is the contour consisting of the semi − circle C_R of radius R together with the real axis from – R to R.

The poles of $f(z) = \frac{1}{z^6+1}$ are the roots of the equation $z^6 + 1 = 0$.

$i.e.,\quad z^6 + 1 = 0 \quad \Rightarrow \quad z = (-1)^{\frac{1}{6}}$

$$\therefore \quad z = (cos\, \pi + +i\, sin\, \pi)^{\frac{1}{6}}$$

$$= [cos\,(2n\pi + \pi) + +i\, sin\,(2n\pi + \pi)]^{\frac{1}{6}}$$

$$= cos\, \frac{(2n+1)\pi}{6} + i\, sin\, \frac{(2n+1)\pi}{6} \qquad \text{(by DeMoivre's theorem)}$$

where $n = 0, 1, 2, 3, 4, 5$.

or $z = e^{i\frac{(2n+1)\pi}{6}}$ where $n = 0, 1, 2, 3, 4, 5$.

or $z = e^{i\pi/6}, e^{3i\pi/6}, e^{5i\pi/6}, e^{7i\pi/6}, e^{9i\pi/6}, e^{11i\pi/6}$

of these poles only $z = e^{i\pi/6}, e^{3i\pi/6}, e^{5i\pi/6}$ lies inside the semi-circle.

$$\therefore \qquad [Res\, f(z)]_{z=e^{i\pi/6}} = \lim_{z \to e^{i\pi/6}} \left[\left(z - e^{i\pi/6}\right)\frac{1}{z^6+1}\right] \qquad \left(= \frac{0}{0}\right)$$

$$= \lim_{z \to e^{i\pi/6}} \left[\frac{1}{6z^5}\right] \quad \text{(By L'Hospital's rule)}$$

$$= \frac{1}{6} e^{-5i\pi/6}$$

Similarly $[Res\, f(z)]_{z=e^{3i\pi/6}} = \dfrac{1}{6} e^{-5i\pi/2}$

and $\quad [Res\, f(z)]_{z=e^{5i\pi/6}} = \dfrac{1}{6} e^{-25i\pi/6}$

hence by residue theorem, we have

$$\int_C f(z)dz = 2\pi i \times (\text{sum of the residues at he poles within } C)$$

$$= \frac{2\pi i}{6} \times \left[e^{-5i\pi/6} + e^{-5i\pi/2} + e^{-25i\pi/6}\right]$$

$$= \frac{\pi i}{3} \times \left[\left(cos\frac{5\pi}{6} - i\, sin\frac{5\pi}{6}\right) + \left(cos\frac{5\pi}{2} - i\, sin\frac{5\pi}{2}\right) + \left(cos\frac{25\pi}{6} - i\, sin\frac{25\pi}{6}\right)\right]$$

$$= \frac{2\pi i}{3}$$

$$i.e., \int_{-R}^{R} f(x)dx + \int_{C_R} f(z)dz = \frac{2\pi}{3}$$

But $\displaystyle\int_{C_R} f(z)dz \to 0$ as $z = Re^{i\theta}$ and $R \to \infty$

Hence $\displaystyle\int_{-\infty}^{\infty} f(x)dx = \frac{2\pi}{3}$

$$i.e., \quad \int_{-\infty}^{\infty} \frac{dx}{x^6+1} = \frac{2\pi}{3} \;\; \text{or} \quad \int_{0}^{\infty} \frac{dx}{x^6+1} = \frac{\pi}{3}$$

EX. 49. Prove that $\displaystyle\int_{-\infty}^{\infty} \frac{x^2}{(x^2+a^2)(x^2+b^2)} dx = \frac{\pi}{a+b} \; (a > 0, b > 0, a \neq b).$

Solution: To evaluate the given integral, consider

$$\int_{-\infty}^{\infty} \frac{z^2}{(z^2+a^2)(z^2+b^2)} dz = \int_C f(z)dz$$

where C is the contour consisting of the semi-circle C_R of radius R together with the part of the real axis from $-R$ to R.

The poles of $f(z)=\dfrac{z^2}{(z^2+a^2)(z^2+b^2)}$ are $z=\pm ia, z=\pm ib$.

But $z=ia$ and $z=ib$ are the only two poles lie in the upper half of the z-plane.

$$\therefore \quad [Res\, f(z)]_{z=ia}=\lim_{z\to ia}(z-ia)f(z)$$

$$=\lim_{z\to ia}(z-ia)\frac{z^2}{(z^2+a^2)(z^2+b^2)}$$

$$=\lim_{z\to ia}\frac{z^2}{(z+ia)(z^2+b^2)}=\frac{-a^2}{(ia+ia)(-a^2+b^2)}=\frac{a}{2i(a^2-b^2)}$$

Also $[Res\, f(z)]_{z=ib}=\lim_{z\to ib}(z-ib)f(z)$

$$=\lim_{z\to ib}(z-ib)\frac{z^2}{(z^2+a^2)(z^2+b^2)}$$

$$=\lim_{z\to ib}\frac{z^2}{(z+ib)(z^2+a^2)}=\frac{-b^2}{(ib+ib)(a^2-b^2)}=\frac{-b}{2i(a^2-b^2)}$$

By Cauchy's Residue theorem, we have

$$\int_C f(z)dz=2\pi i\times(\text{sum of the residues at he poles within } C)$$

$$\therefore \quad \int_C f(z)dz=2\pi i\left[\frac{a}{2i(a^2-b^2)}-\frac{b}{2i(a^2-b^2)}\right]$$

$$=\pi\left[\frac{a-b}{a^2-b^2}\right]=\frac{\pi}{a+b}$$

$$i.e., \int_{-R}^{R} f(x)dx+\int_{C_R} f(z)dz=\frac{\pi}{a+b} \quad \text{But} \int_{C_R} f(z)dz \to 0 \text{ as } z=Re^{i\theta} \text{ and } R\to\infty$$

$$\text{Hence} \quad \int_{-\infty}^{\infty} f(x)dx=\int_{-\infty}^{\infty}\frac{x^2}{(x^2+a^2)(x^2+b^2)}dx=\frac{\pi}{a+b}$$

EX. 50. Prove that $\displaystyle\int_{-\infty}^{\infty}\frac{x^2}{(x^2+1)(x^2+4)}dx$ using residue theorem.

Solution: To evaluate the given integral, consider

$$\int_{-\infty}^{\infty}\frac{z^2}{(z^2+1)(z^2+4)}dz=\int_C f(z)dz$$

where C is the contour consisting of the semi-circle C_R of radius R together with the part of the real axis from $-R$ to R.

The poles of $f(z) = \dfrac{z^2}{(z^2+1)(z^2+4)}$ are $z = \pm i, z = \pm 2i$.

But $z = i$ and $z = 2i$ are the only two poles lie inside C.

$$\therefore \quad [Res\, f(z)]_{z=i} = \lim_{z\to i}(z-i)f(z)$$

$$= \lim_{z\to i}(z-i)\frac{z^2}{(z^2+1)(z^2+4)}$$

$$= \lim_{z\to i}\frac{z^2}{(z+i)(z^2+4)} = \frac{-1}{(i+i)(-1+4)} = \frac{-1}{6i}$$

Also $[Res\, f(z)]_{z=2i} = \lim_{z\to 2i}(z-2i)f(z)$

$$= \lim_{z\to 2i}(z-2i)\frac{z^2}{(z^2+1)(z^2+4)}$$

$$= \lim_{z\to 2i}\frac{z^2}{(z^2+1)(z+2i)} = \frac{-4}{(-4+1)(2i+2i)} = \frac{1}{3i}$$

By Cauchy's Residue theorem, we have

$$\int_C f(z)dz = 2\pi i \times (\text{sum of the residues at he poles within } C)$$

$$\therefore \quad \int_C f(z)dz = 2\pi i\left[\frac{-1}{6i} + \frac{1}{3i}\right] = \frac{\pi}{3}$$

$$i.e., \int_{-R}^{R} f(x)dx + \int_{C_R} f(z)dz = \frac{\pi}{3} \quad \text{But} \int_{C_R} f(z)dz \to 0 \text{ as } z = Re^{i\theta} \text{ and } R \to \infty$$

Hence $\displaystyle\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)(x^2+4)}dx = \frac{\pi}{3}$

EX. 51. Evaluate by contour integration $\displaystyle\int_0^\infty \frac{dx}{1+x^2}$.

Solution: Consider $\displaystyle\int_0^\infty \frac{dx}{1+x^2} = \int_C f(z)dz$

where C is the contour consisting of the semi-circle C_R of radius R together with the part of the real axis from $-R$ to R.

The integrand has simple poles at $z = \pm i$. The pole $z = i$ is inside C and $z = -i$ is outside C.

$$[Res\, f(z)]_{z=i} = \lim_{z\to i}(z-i)f(z) = \lim_{z\to i}\frac{1}{z+i} = \frac{1}{2i}$$

Hence by Cauchy's residue theorem,

$$\int_C f(z)dz = 2\pi i \times \frac{1}{2i} = \pi$$

$$i.e., \int_{-R}^{R} f(x)dx + \int_{C_R} f(z)dz = \pi$$

But $\int_{C_R} f(z)dz \to 0$ as $z = Re^{i\theta}$ and $R \to \infty$

Hence $\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{\infty} \frac{1}{(x^2+1)}dx = \pi$

or $\int_{0}^{\infty} \frac{1}{x^2+1}dx = \frac{\pi}{2}$

EX. 52. Prove that $\int_{-\infty}^{\infty} \frac{x^2-x+2}{x^4+10x^2+9}dx = \frac{5\pi}{12}$.

Solution: To evaluate the given integral, we consider

$$\int_C \frac{z^2-z+2}{z^4+10z^2+9}dz = \int_C \frac{z^2-z+2}{(z^2+1)(z^2+9)}dz = \int_C f(z)\,dz$$

where C is the contour consisting of the semi-circle C_R of radius R together with the part of the real axis from $-R$ to R. Observe that the integrand has simple poles at $z = \pm i, z = \pm 3i$. But $z = i$ and $z = 3i$ are the only poles lie inside the semi-circle of the contour C.

$\therefore$ By Residue theorem, we have

$$\int_C f(z)dz = 2\pi i\{[Res\, f(z)]_{z=i} + [Res\, f(z)]_{z=3i}\}$$

$$= 2\pi i\left[\lim_{z\to i}(z-i)f(z) + \lim_{z\to 3i}(z-3i)f(z)\right]$$

$$= 2\pi i\left[\lim_{z\to i}\frac{z^2-z+2}{(z+i)(z^2+9)} + \lim_{z\to 3i}\frac{z^2-z+2}{(z^2+1)(z+3i)}\right]$$

$$= 2\pi i\left[\frac{i^2 - i + 2}{(i+i)(i^2+9)} + \frac{9i^2 - 3i + 2}{(9i^2+1)(3i+3i)}\right]$$

$$= 2\pi i\left[\frac{-i+1}{16i} + \frac{-3i-7}{-48i}\right] = \frac{5\pi}{12}$$

$$i.e., \int_{-R}^{R} f(x)dx + \int_{C_R} f(z)dz = \frac{5\pi}{12}$$

Taking $R \to \infty$, $\int_{-\infty}^{\infty} f(x)dx = \frac{5\pi}{12}$ where $\lim_{R\to\infty} \int_{C_R} f(z)dz = 0$

Hence $\int_{-\infty}^{\infty} \frac{x^2 - x + 2}{x^4 + 10x^2 + 9} dx = \frac{5\pi}{12}$.

EX. 53. Use the method of contour integration to evaluate $\int_{-\infty}^{\infty} \frac{x^2}{(x^2+a^2)^3} dx$.

Solution: Let $f(z) = \frac{z^2}{(z^2+a^2)^3}$

Consider $\int_C f(z)dz$ where C is a closed contour consisting of the upper half C_R of a large circle $|z| = R$ and the real axis from $-R$ to R.

Poles of $f(z)$ are given by $(z^2 + a^2)^3 = 0$ or $z^2 = -a^2 \Rightarrow z = \pm ia$

Evidently $z = ia$ is the only pole of order 3 lie within C.

To get residue at $z = ia$, put $z = ia + w$ in $f(z)$. Then

$$f(ia + w) = \frac{(ia+w)^2}{[(ia+w)^2 + a^2]^3} = \frac{w^2 - a^2 + 2iaw}{(w^2 + 2iaw)^3}$$

$$= \frac{w^2 - a^2 + 2iaw}{(2iaw)^3}\left[1 + \frac{w}{2ia}\right]^{-3}$$

$$= -\frac{1}{8i}.\frac{w^2 - a^2 + 2iaw}{a^3w^3}\left[1 - \frac{3w}{2ia} + \frac{6w^2}{4i^2a^2} - \cdots\right]$$

$$= \frac{w^2 - a^2 + 2iaw}{-8ia^3}\left[\frac{1}{w^3} - \frac{3}{2iaw^2} - \frac{3}{2a^2w} - \cdots\right]$$

Hence residue = coefficient of $\frac{1}{w}$

$$= -\frac{1}{8ia^3}\left[1 + \frac{3}{2} - 3\right] = \frac{1}{16ia^3}$$

Hence by Cauchy's Residue theorem, we have

$$\int_C f(z)dz = 2\pi i \times \text{sum of the residues within } C$$

$$i.e., \quad \int_{-R}^{R} f(x)dx + \int_{C_R} f(z)dz = 2\pi i \times \frac{1}{16ia^3}$$

$$i.e., \quad \int_{-R}^{R} \frac{x^2}{(x^2+a^2)^3}dx + \int_{C_R} \frac{z^2}{(z^2+a^2)^3}dz = \frac{\pi}{8a^3} \qquad (1)$$

$$\text{Now} \quad \left|\int_{C_R} \frac{z^2}{(z^2+a^2)^3}dz\right| \le \int_{C_R} \frac{|z|^2}{(|z^2+a^2|)^3}dz$$

$$\le \frac{R^2}{(R^2-a^2)^3}\int_0^{\pi} Rd\theta \ \left[\because z = Re^{i\theta}, |dz| = Rd\theta\right]$$

$$= \frac{R^2\pi}{(R^2-a^2)^3} \text{ and this } \to 0 \text{ as } R \to \infty$$

$$\therefore \int_{C_R} \frac{z^2}{(z^2+a^2)^3}dz = 0 \text{ as } R \to \infty$$

Hence by making $R \to \infty$, equation (1)becomes $\displaystyle\int_{-\infty}^{\infty} \frac{x^2}{(x^2+a^2)^3}dx = \frac{\pi}{8a^3}$

EX. 54. Evaluate $\displaystyle\int_0^{\infty} \frac{dx}{(x^2+9)(x^2+4)^2}$ using residue theorem.

Solution: Here we consider $\displaystyle\int_0^{\infty} \frac{dz}{(z^2+9)(z^2+4)^2} = \int_C f(z)dz$

where C is the contour consisting of the semi-circle C_R of radius R and the segment of the real axis from – R to R.

For the function $f(z)$, $z = \pm 3i$ are two simple poles and $z = \pm 2i$ are two poles of second order. Of these four poles, only $z = 2i$ and $z = 3i$ are inside C.

$$[Res\ f(z)]_{z=2i} = \frac{1}{(2-1)!}\lim_{z\to 2i}\frac{d}{dz}[(z-2i)^2 f(z)]$$

$$= \lim_{z\to 2i}\frac{d}{dz}\left[\frac{1}{(z^2+9)(z+2i)^2}\right]$$

$$= \lim_{z\to 2i}\left[\frac{-2(2z^2+2zi+9)}{(z^2+9)^2(z+2i)^3}\right] = \frac{3i}{800}$$

$$[Res\, f(z)]_{z=3i} = \lim_{z\to 3i}[(z-3i)f(z)]$$

$$= \lim_{z\to 2i}\left[\frac{1}{(z+3i)(z^2+4)^2}\right]$$

$$= \frac{-i}{150}$$

Hence by Cauchy's Residue theorem, we have

$$\int_C f(z)dz = 2\pi i \times \text{sum of the residues within } C$$

$$i.e., \int_{-R}^{R} f(x)dx + \int_{C_R} f(z)dz = 2\pi i \times \left(\frac{3i}{800} - \frac{i}{150}\right) = \frac{7\pi}{1200} \qquad (1)$$

Hence by making $R \to \infty$, equation (1) becomes

$$\int_{-\infty}^{\infty} f(x)dx + \lim_{R\to\infty}\int_{C_R} f(z)dz = \frac{7\pi}{1200}$$

when $R \to \infty, |z| \to \infty, \therefore \int_{C_R} f(z)dz = 0$

$$\text{Thus } \int_{-\infty}^{\infty} f(x)dx = \frac{7\pi}{1200}$$

$$i.e., \quad \int_{-\infty}^{\infty} \frac{dx}{(x^2+9)(x^2+4)^2} = \frac{7\pi}{1200}$$

$$\text{or} \quad 2\int_{0}^{\infty} \frac{dx}{(x^2+9)(x^2+4)^2} = \frac{7\pi}{1200}$$

$$\text{or} \quad \int_{0}^{\infty} \frac{dx}{(x^2+9)(x^2+4)^2} = \frac{7\pi}{2400}$$

EX. 55. Prove that $\int_{-\infty}^{\infty} \frac{x^2-x+2}{x^4+10x^2+9}dx = \frac{5\pi}{12}$.

Solution: To evaluate the given integral, we consider

$$\int_{-\infty}^{\infty} \frac{z^2-z+2}{z^4+10z^2+9}dz = \int_{0}^{\infty} \frac{z^2-z+2}{(z^2+1)(z^2+9)}dz = \int_C f(z)dz$$

where C is the contour consisting of the semi-circle C_R of radius R together with the part of the real axis from – R to R. Observe that the integrand has simple poles at $z = \pm i$ and

$z = \pm 3i$. But $z = i$ and $z = 3i$ are the only two poles lie inside the semi-circle of the contour C

For the function $f(z)$, $z = \pm 3i$ are two simple poles and $z = \pm 2i$ are two poles of second order. Of these four poles, only $z = 2i$ and $z = 3i$ are inside C.

Hence by Cauchy's Residue theorem, we have

$$\int_C f(z)dz = 2\pi i \times \text{sum of the residues within } C$$

$$= 2\pi i\{[Res\, f(z)]_{z=i} + [Res\, f(z)]_{z=3i}\}$$

$$= 2\pi i\left[\lim_{z\to i}(z-i)f(z) + \lim_{z\to 3i}(z-3i)f(z)\right]$$

$$= 2\pi i\left[\lim_{z\to i}\frac{z^2-z+2}{(z+i)(z^2+9)} + \lim_{z\to 3i}\frac{z^2-z+2}{(z+i)(z+3i)}\right]$$

$$= 2\pi i \times \frac{10}{48i} = \frac{5\pi}{12}$$

$$i.e., \quad \int_{-R}^{R} f(x)dx + \int_{C_R} f(z)dz = \frac{5\pi}{12} \qquad (\because \text{ on real axis } z = x)$$

Taking $R \to \infty$, $\displaystyle\int_{-\infty}^{\infty} f(x)dx = \frac{5\pi}{12}$ where $\displaystyle\lim_{R\to\infty}\int_{C_R} f(z)dz = 0$

Hence $\displaystyle\int_{-\infty}^{\infty}\frac{x^2-x+2}{x^4+10x^2+9}dx = \frac{5\pi}{12}$

Note: Instead of proving separately that $\displaystyle\int_{C_2} Q(z)dz = 0$ at $R \to \infty$, we can remark that $Q(z)$ satisfies the conditions of the theorem 8 and start directly from equation (2).

2.21. Evaluation of certain improper integrals involving trigonometric functions:

Consider the product $e^{imz}Q(z)$ where $m > 0$ and $Q(z)$ satisfies the conditions of the theorem in 8.

$$\text{Now} \qquad |e^{imz}| = |e^{im(x+iy)}|$$
$$= |e^{imx}.e^{-my}|$$
$$= e^{-my}$$
$$< 1, \text{for } y > 0$$
$$\text{Therefore } |e^{imz}Q(z)| = |e^{imx}|.|Q(z)|$$
$$< Q(z)$$

Since $\int_{C_2} e^{imz}Q(z) \to 0$ where C_2 is the semicircle of the above figure, it follows that

$$\int_{C_2} e^{imz}Q(z) = 0$$

Hence the conclusions of theorem 8 can be applied for $e^{imz}Q(z)$.

So we have the following result:

$$\int_{-\infty}^{\infty} e^{imz}Q(z)\,dz = 2\pi i \sum \text{residues of } e^{imz}Q(z) \text{ at its poles in the upper half plane.}$$

On taking the real and imaginary parts of this result, we see that by this method we can evaluate integrals of the type

$$\int_{-\infty}^{\infty} f(x)\ \cos mx\, dx \text{ and } \int_{-\infty}^{\infty} f(x)\ \sin mx\, dx$$

EX. 56. Evaluate by contour integration $\int_0^{\infty} \frac{\cos mx}{a^2 + x^2} dx$.

Solution: Here we consider

$$\int_C \frac{e^{imz}}{a^2 + z^2} dz = \int_C Q(z)\,dz$$

where C is the contour consisting of the semicircle C_2 of radius R and C_1 is the segment of the real axis from $-R$ to R as shown in the above figure.

$$\text{Then} \int_{-R}^{R} Q(x)\,dx + \int_{C_2} Q(z)\,dz = 2\pi i \sum \text{reidues of } Q(z) \text{ in the upper half plane} \qquad (1)$$

Now $|z| = R$ on C_2.

$$|z^2 + a^2| \geq |z|^2 - a^2$$
$$\geq R^2 - a^2$$

Therefore $$\left|\frac{1}{z^2 + a^2}\right| \leq \frac{1}{R^2 - a^2}$$

$$|e^{imz}| = |e^{im(x+iy)}| = |e^{imx}.e^{-imy}|$$
$= e^{-my} < 1$ since $y > 0$ in the upper half plane.

Therefore $\left|\frac{e^{imz}}{z^2+a^2}\right| = |e^{imz}|.\left|\frac{1}{z^2+a^2}\right|$

$$< \frac{1}{R^2 - a^2}$$

Hence $$\left|\int_{C_2} \frac{e^{imz}}{z^2 + a^2} dz\right| < \int_{C_2} \frac{1}{R^2 - a^2} |dz|$$

$$< \frac{\pi.R}{R^2 - a^2}$$

and this approaches zero as $R \to 0$.

Therefore $\int_{C_2} Q(z)\, dz = 0$ in the limit $R \to \infty$.

Hence taking limits in (1), as $R \to \infty$, we have

$$\int_{-\infty}^{\infty} Q(x)\, dx = 2\pi i \sum \text{reidues of}\, Q(z) (2)$$

To get the residues of $Q(z)$, we solve $z^2 + a^2 = 0$.

This gives $z = \pm ia$. The only pole in the upper half of the z-plane is ia.

Residue at $z = ia$ is $= \lim_{z \to ia} \frac{(z - ia)e^{imz}}{z^2 + a^2}$

$$= \lim_{z \to ia} \frac{e^{imz}}{z + ia}$$

$$= \frac{e^{im.ia}}{ia + ia} = \frac{e^{-ma}}{2ia} \qquad (3)$$

Substituting (3) in (2), we get

$$\int_{-\infty}^{\infty} Q(x)\, dx = 2\pi i \times \frac{e^{-ma}}{2ia} = \frac{\pi}{a} e^{-ma}$$

$$i.e., \qquad \int_{-\infty}^{\infty} \frac{e^{imx}}{x^2 + a^2} dx = \frac{\pi}{a} e^{-ma} \qquad (4)$$

Equating the real part of both sides,

$$\int_{-\infty}^{\infty} \frac{cos\ mx}{a^2+x^2} dx = \frac{\pi}{a} e^{-ma}$$

Since $\frac{cos\ mx}{a^2+x^2}$ is an even function of x, we have

$$\int_{-\infty}^{\infty} \frac{cos\ mx}{a^2+x^2} dx = 2\int_{0}^{\infty} \frac{cos\ mx}{a^2+x^2} dx = \frac{\pi}{a} e^{-ma}$$

Therefore $$\int_{0}^{\infty} \frac{cos\ mx}{a^2+x^2} dx = \frac{\pi}{2a} e^{-ma}$$

EX. 57. Prove that $\int_{-\infty}^{\infty} \frac{cos\ ax}{x^2+1} dx = \pi e^{-a}, a \geq 0.$

Solution: We know that $cos\ ax$ is the real part of e^{iax}.

$\therefore$ We consider the function $f(z) = \frac{e^{iaz}}{z^2+1}$.

Now, the poles of $f(z)$ are given by $z = \pm i$, but $z = i$ is the only pole lie in the upper half of the z-plane.

$$\therefore \quad [Res\ f(z)]_{z=i} = \lim_{z\to i}(z-i)f(z) = \lim_{z\to i} \frac{e^{iaz}}{z+i} = \frac{e^{-a}}{2i}$$

Thus $$\int_{C} \frac{e^{iaz}}{z^2+1} dz = 2\pi i \left(\frac{e^{-a}}{2i}\right) = \pi e^{-a} \qquad (1)$$

where C is the contour consisting of the semi-circle C_R of radius R together with the part of the real axis from $-R$ to R.

On the semi-circle $C_R, z = Re^{i\theta}$

Observe that $\lim_{R\to\infty} \int_{C_R} f(z)dz = 0$

$\therefore$ From (1), we get $\int_{-\infty}^{\infty} f(x)dx = \pi e^{-a}$

Now, equating the real parts, we get $\int_{-\infty}^{\infty} \frac{cos\ ax}{x^2+1} dx = \pi e^{-a}$.

EX. 58. Evaluate $\int_0^\infty \frac{cos\ x}{(x^2+1)^2}dx$.

Solution: Let $\int_0^\infty \frac{cos\ x}{(x^2+1)^2}dx = \int_0^\infty f(x)dx$

where $\quad f(x) = \frac{cos\ x}{(x^2+1)^2} = \text{R. P. of } \frac{e^{ix}}{(x^2+1)^2}$

Consider the integral

$$\int_C f(z)dz = \int_C \frac{e^{iz}}{(z^2+1)^2}dz \qquad (1)$$

where C is the contour consisting of the semi-circle C_R of radius R together with the part of the real axis from $-R$ to R.

From (1), $\int_C f(z)dz = \int_{-R}^{R} f(x)dx + \int_{C_R} f(z)dz \quad$ (on the real axis, $z = x$) $\qquad$ (2)

Now the poles of $f(z)$ are given by $z^2+1=0, i.e., z=\pm i$

Of these poles only the pole $z = i$ of order 2 lie inside the upper half of the plane (i.e., the circle C).

$$\therefore \quad [Res\ f(z)]_{z=i} = \frac{1}{(2-1)!}\lim_{z\to i}\left[\frac{d}{dz}(z-i)^2 f(z)\right]$$

$$= \lim_{z\to i}\frac{d}{dz}\left[(z-i)^2\frac{e^{iz}}{(z+i)^2(z-i)^2}\right]$$

$$= \lim_{z\to i}\frac{d}{dz}\left[\frac{e^{iz}}{(z+i)^2}\right] = -\frac{i}{2e}$$

By Residue theorem, we have

$$\int_C f(z)dz = 2\pi i \times \text{sum of residues of } f(z) \text{ at the poles within } C$$

$$= 2\pi i\left(-\frac{i}{2e}\right) = \frac{\pi}{e} \qquad (3)$$

As $R\to\infty$, for any point z on the semi-circle C_R, $|z|\to\infty, i.e., f(z)\to 0$

$$\therefore \quad \lim_{|z|\to\infty}\int_{C_R} f(z)dz \to 0 \qquad (4)$$

From (2), (3) and (4), we get

$$\int_{-\infty}^{\infty} f(x)dx = \frac{\pi}{e}$$

$i.e.,$ $$\int_{-\infty}^{\infty} \frac{e^{ix}}{(x^2+1)^2}dx = \frac{\pi}{e}$$

Equating real parts both sides, we get

$$\int_{-\infty}^{\infty} \frac{\cos x}{(x^2+1)^2}dx = \frac{\pi}{e}$$

$\therefore$ $$\int_{0}^{\infty} \frac{\cos x}{(x^2+1)^2}dx = \frac{\pi}{2e}$$

EX. 59. Evaluate $\int_0^{\infty} \frac{x \sin mx}{x^4+16}dx$ using residue theorem.

Solution: Consider the integral $\int_C \frac{z \sin mz}{z^4+16}dz = \int_C f(z)dz$

where C is the contour consisting of the semi-circle C_R of radius R and bounding diameter – R to R.

We know that sin mz is the imaginary part of e^{imz}.

$\therefore$ Take $f(z) = \frac{z\, e^{imz}}{z^4+16}$

Poles of $f(z)$ are given by $z^4 + 16 = 0$

$i.e.,$ $z^4 = -16 = 16(\cos\pi + i\sin\pi)$

or $z^4 = 2^4[\cos(2n+1)\pi + i\sin(2n+1)\pi]$

or $z = 2\left[\cos\frac{(2n+1)\pi}{4} + i\sin\frac{(2n+1)\pi}{4}\right]$ where $n = 0, 1, 2, 3$

If $n = 0, z_1 = 2\left[\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right] = 2\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right) = \sqrt{2} + i\sqrt{2}$

If $n = 1, z_1 = 2\left[\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right] = 2\left(-\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right) = -\sqrt{2} + i\sqrt{2}$

If $n = 2, z_1 = 2\left[\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right] = 2\left(-\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right) = -\sqrt{2} - i\sqrt{2}$

If $n = 3, z_1 = 2\left[\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}\right] = 2\left(\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right) = \sqrt{2} - i\sqrt{2}$

Of these poles, only the two poles $z_1 = 2e^{i\frac{\pi}{4}} = \sqrt{2} + i\sqrt{2}$ and $z_2 = 2e^{i\frac{3\pi}{4}} = -\sqrt{2} + i\sqrt{2}$ lie within the circle C.

Hence we have to calculate the corresponding residues.

$$[Res\, f(z)]_{z=z_1} = \lim_{z\to z_1}(z - z_1)\, f(z)$$

$$= \lim_{z\to z_1}(z - z_1)\frac{z\, e^{imz}}{z^4 + 16} \quad \left(=\frac{0}{0}\right). \text{Applying L Hospital's rule}$$

$$= \lim_{z\to z_1}\frac{(z - z_1)\left\{e^{imz} + z.im.e^{imz}\right\} + ze^{imz}}{4z^3}$$

$$= \frac{z_1\, e^{imz_1}}{4z_1^3} = \frac{e^{imz_1}}{4z_1^2} = \frac{e^{im(\sqrt{2}+i\sqrt{2})}}{4\left(\sqrt{2} + i\sqrt{2}\right)^2} \qquad (1)$$

$$= \frac{e^{im\sqrt{2}-m\sqrt{2}}}{8(1+i)^2} = \frac{e^{-m\sqrt{2}}e^{im\sqrt{2}}}{8(2i)} = \frac{i\, e^{-m\sqrt{2}}\left(cos\, m\sqrt{2} + i\, sin\, m\sqrt{2}\right)}{-16}$$

Similarly $[Res\, f(z)]_{z=z_2} = \lim_{z\to z_2}(z - z_2)\, f(z)$

$$= \frac{e^{im(-\sqrt{2}+i\sqrt{2})}}{4\left(\sqrt{2} + i\sqrt{2}\right)^2} \quad \text{(from (1))}$$

$$= \frac{i\, e^{-m\sqrt{2}}\left(cos\, m\sqrt{2} - i\, sin\, m\sqrt{2}\right)}{16}$$

By Residue theorem,

$$\int_C f(z)dz = 2\pi i \times \text{sum of residues of } f(z) \text{ at the poles within } C$$

$$= 2\pi i\left[\frac{i\, e^{-m\sqrt{2}}\left(cos\, m\sqrt{2} + i\, sin\, m\sqrt{2}\right)}{-16} + \frac{i\, e^{-m\sqrt{2}}\left(cos\, m\sqrt{2} - i\, sin\, m\sqrt{2}\right)}{16}\right]$$

$$= 2\pi i\left(-\frac{i}{16}e^{-m\sqrt{2}}\right)\left[\left(cos\, m\sqrt{2} + i\, sin\, m\sqrt{2}\right) - \left(cos\, m\sqrt{2} - i\, sin\, m\sqrt{2}\right)\right]$$

$$= \frac{\pi}{8}e^{-m\sqrt{2}}\left(2i\, sin\, m\sqrt{2}\right)$$

$$= i\frac{\pi}{4}e^{-m\sqrt{2}}sin\, m\sqrt{2}$$

$$i.e., \int_{-R}^{R} f(x)dx + \int_{C_R} f(z)dz = i\frac{\pi}{4}e^{-m\sqrt{2}}sin\, m\sqrt{2} \qquad (2)$$

On the semi – circle $C_R, z = Re^{i\theta}$, we observe that $\int_{C_R} f(z)dz \to 0$ as $R \to \infty$.

∴ From (2), we get

$$\int_{-\infty}^{\infty} f(x)dx = i\frac{\pi}{4}e^{-m\sqrt{2}} sin\, m\sqrt{2}$$

$$\int_{-\infty}^{\infty} \frac{x\, e^{imx}}{x^4+16}dx = i\frac{\pi}{4}e^{-m\sqrt{2}} sin\, m\sqrt{2}$$

Now equating the imaginary parts, we get

$$\int_{-\infty}^{\infty} \frac{x\, sin\, mx}{x^4+16}dx = \frac{\pi}{4}e^{-m\sqrt{2}} sin\, m\sqrt{2}$$

$i.e.,$
$$2\int_{0}^{\infty} \frac{x\, sin\, mx}{x^4+16}dx = \frac{\pi}{4}e^{-m\sqrt{2}} sin\, m\sqrt{2}$$

or
$$\int_{0}^{\infty} \frac{x\, sin\, mx}{x^4+16}dx = \frac{\pi}{8}e^{-m\sqrt{2}} sin\, m\sqrt{2}$$

Note: $\int_{0}^{\infty} \frac{x\, sin\, mx}{x^4+a^4}dx = \frac{\pi}{2a^2}e^{-\frac{am}{\sqrt{2}}} sin\left(\frac{am}{\sqrt{2}}\right)$

EX.60. Show by the method of contour integration that

$$\int_{0}^{\infty} \frac{cos\, mx}{(x^2+a^2)^2}dx = \frac{\pi}{4a^3}(1+ma)e^{-ma} \quad (a>0, b>0)$$

Solution: Let $\int_{0}^{\infty} \frac{cos\, mx}{(x^2+a^2)^2}dx = \int_{0}^{\infty} f(x)dx$

where $f(x) = \frac{cos\, mx}{(x^2+a^2)^2} =$ Real part of $\frac{e^{imx}}{(x^2+a^2)^2}$

Consider the integral $\int_C f(z)dz = \int_C \frac{e^{imz}}{(z^2+a^2)^2}dz$

where C is the closed contour consisting of the semi-circle C_R: $|z| = R$ and real axis from $-R$ to R.

$$\therefore \int_C f(z)dz = \int_{-R}^{R} f(x)dx + \int_{C_R} f(z)dz \qquad \text{(on real axis } z = x) \qquad (1)$$

Evidently $\lim_{|z|\to a} \frac{1}{(z^2+a^2)^2} = 0$

Hence $\lim_{R\to\infty}\int_{C_R}\frac{e^{imz}}{(z^2+a^2)^2}dz = 0$ or $\lim_{R\to\infty}\int_{C_R} f(z)dz = 0$ (2)

Now the poles of $f(z)$ are given by $z^2+a^2=0, i.e., z=\pm ia$

Of these poles only the pole $z=ia$ of order 2 lie inside the circle C.

$$[Res\, f(z)]_{z=ia} = \frac{1}{(2-1)!}\lim_{z\to ia}\left[\frac{d}{dz}(z-ia)^2 f(z)\right]$$

$$= \lim_{z\to ia}\left[\frac{d}{dz}\frac{e^{imz}}{(z+ia)^2}\right]$$

$$= \lim_{z\to ia}\frac{(z+ia)^2.\, im.\, e^{imz} - e^{imz}.\,2(z+ia)}{(z+ia)^4}$$

$$= \lim_{z\to ia}\frac{e^{imz}(z+ia)[im(z+ia)-2]}{(z+ia)^4}$$

$$= \lim_{z\to ia}\frac{e^{imz}[im(z+ia)-2]}{(z+ia)^3} \qquad = \frac{e^{-ma}(1+ma)}{4a^3 i}$$

By Cauchy's Residue theorem

$$\int_C f(z)dz = 2\pi i \times \text{sum of the residues}$$

$$i.e., \qquad \int_{-R}^{R} f(x)dx + \int_{C_R} f(z)dz = \frac{\pi}{2a^3}(1+ma)e^{-ma}, \text{ from (1)}$$

Making $R\to\infty$ and noting (2), we get

$$\int_{-\infty}^{\infty} f(x)dx = \frac{\pi}{2a^3}(1+ma)e^{-ma}$$

$$\text{or} \quad \int_{-\infty}^{\infty}\frac{e^{imx}}{(x^2+a^2)^2}dx = \frac{\pi}{2a^3}(1+ma)e^{-ma}$$

equating real parts from both sides,

$$\int_{-\infty}^{\infty}\frac{cos\, mx}{(x^2+a^2)^2}dx = \frac{\pi}{2a^3}(1+ma)e^{-ma}$$

$$\text{or} \quad 2\int_{0}^{\infty}\frac{cos\, mx}{(x^2+a^2)^2}dx = \frac{\pi}{2a^3}(1+ma)e^{-ma}$$

$$\text{or} \quad \int_{0}^{\infty}\frac{cos\, mx}{(x^2+a^2)^2}dx = \frac{\pi}{4a^3}(1+ma)e^{-ma}$$

EX. 61. Evaluate $\int_{-\infty}^{\infty} \frac{\cos x}{x^2 + a^2} dx \quad (a > 0)$ using residue theorem.

Solution: Let $f(z) = \frac{e^{iz}}{z^2 + a^2}$

Consider $\int_C f(z)dz$ where C is the closed contour as shown in the figure.

Poles of $f(z)$ are given by $z^2 + a^2 = 0$ or $z = \pm ia$.

But $z = ia$ only lies inside C.

$$[Res\, f(z)]_{z=ia} = \lim_{z \to ia} (z - ia) f(z)$$

$$= \lim_{z \to ia} \frac{e^{iz}}{z + ia} = \frac{1}{2ia} e^{-a}$$

∴ By Residue theorem,

$$\int_C f(z)dz = 2\pi i \times sum\ of\ the\ residues$$

$$= 2\pi i \times \frac{1}{2ia} e^{-a} = \frac{\pi e^{-a}}{a}$$

$$i.e., \int_{-R}^{R} f(x)dx + \int_{C_R} f(z)dz = \frac{\pi e^{-a}}{a}$$

$$i.e., \int_{-R}^{R} \frac{e^{ix}}{x^2 + a^2} dx + \int_{C_R} \frac{e^{iz}}{z^2 + a^2} dz = \frac{\pi e^{-a}}{a}$$

Making $R \to \infty$, we get

$$\int_{-\infty}^{\infty} \frac{e^{ix}}{x^2 + a^2} dx = \frac{\pi e^{-a}}{a} \qquad \left[since \lim_{R \to \infty} \int_{C_R} \frac{e^{iz}}{z^2 + a^2} dz = 0\right]$$

Equating real parts, we get

$$\int_{-\infty}^{\infty} \frac{\cos x}{x^2 + a^2} dx = \frac{\pi e^{-a}}{a}$$

8. LECTURE NOTES

Solutions of Algebraic and Transcendental equations:

1) **Polynomial function:** A function $f(x)$ is said to be a polynomial function

if $f(x)$ is a polynomial in x.

ie, $f(x) = a_0x^n + a_1x^{n-1} + \cdots \ldots \ldots \ldots .. + a_{n-1}x + a_n$

where $a_0 \neq 0$, the co-efficients $a_0, a_1 \ldots\ldots\ldots a_n$ are real constants and n is a

non-negative integer.

2) **Algebraic function:** A function which is a sum (or) difference (or) product of two polynomials is called an algebraic function. Otherwise, the function is called a transcendental (or) non-algebraic function.

Eg: (i) $f(x) = c_1e^x + c_2e^{-x} = 0$ (ii) $f(x) = e^{5x} - \frac{x^3}{2} + 3 = 0$

3) **Root of an equation:** A number α is called a root of an equation $f(x)=0$ if $f(\alpha)=0$. We also say that α is a zero of the function.

Note: The roots of an equation are the abscissae of the points where the graph $y = f(x)$ cuts the x-axis.

Methods to find the roots of *f* (x) = 0

Direct method:

We know the solution of the polynomial equations such as linear equation $ax + b$ =0, and quadratic equation $ax^2 + bx + c = 0$,using direct methods or analytical methods. Analytical methods for the solution of cubic and quadratic equations are also available.

1.1. Bisection method:

Bisection method is a simple iteration method to solve an equation. This method is also known as Bolzano method of successive bisection. Sometimes it is referred to as half-interval method.

(i) Suppose we know an equation of the form $f(x)=0$ has exactly one real root between two real numbers x_0, x_1 .The number is chosen such that $f(x_0)$ and $f(x_1)$ will have opposite sign.

(ii) Let us bisect the interval $[x_0, x_1]$ into two half intervals and find the mid point $x_2 = \frac{x_0 + x_1}{2}$. If $f(x_2)=0$ then x_2 is a root.

(iii) If $f(x_1)$ and $f(x_2)$ have same sign then the root lies between x_0 and x_2.

(iv) The interval is taken as $[x_0, x_2]$. Otherwise the root lies in the interval $[x_2, x_1]$.

(v) Next calculate x 3 , x4, x5---,until two consecutive iterations are equal. Then we stop the process after getting desired accuracy.

This method is known as Bisection Method

PROBLEMS

1). Find a root of the equation $x^3 - 5x + 1 = 0$ using the bisection method in 5 – stages

Sol Let $f(x) = x^3 - 5x + 1$. We note that $\begin{matrix} f(0) > 0 \\ f(1) < 0 \end{matrix}$ $\quad and$

$\therefore$ One root lies between 0 and 1

Consider $x_0 = 0 \;\; and \;\; x_1 = 1$

By Bisection method the next approximation is

$$x_2 = \frac{x_0 + x_1}{2} = \frac{1}{2}(0+1) = 0.5$$
$$\Rightarrow f(x_2) = f(0:5) = -1.375 < 0 \;\; and \;\; f(0) > 0$$

We have the root lies between 0 and 0.5

Now $x_3 = \frac{0+0.5}{2} = 0.25$

We find $f(x_3) = -0.234375 < 0 \quad and \quad f(0) > 0$

Since $f(0) > 0$, we conclude that root lies between $x_0 \ and \ x_3$

The third approximation of the root is

$x_4 = \frac{x_0+x_3}{2} = \frac{1}{2}(0 + 0.25) = 0.125$

We have $f(x_4) = 0.37495 > 0$

Since $f(x_4) > 0 \ and \ f(x_3) < 0$, the root lies between

$x_4 = 0.125 \ and \ x_3 = 0.25$

Considering the 4th approximation of the roots

$x_5 = \frac{x_3 + x_4}{2} = \frac{1}{2}(0.125 + 0.25) = 0.1875$

$f(x_5) = 0.06910 > 0$, since $f(x_5) > 0 \ and \ f(x_3) < 0$ the root must lie between

$x_5 = 0.18758 \ and \ x_3 = 0.25$

Here the fifth approximation of the root is

$$x_6 = \frac{1}{2}(x_5 + x_3)$$
$$= \frac{1}{2}(0.1875 + 0.25)$$
$$= 0.21875$$

We are asked to do up to 5 stages

We stop here 0.21875 is taken as an approximate value of the root

and it lies between 0 and 1

2) Find a root of the equation $x^3 - 4x - 9 = 0$ using bisection method in four stages

Sol Let $f(x) = x^3 - 4x - 9$

We note that $f(2) < 0$ and $f(3) > 0$

$\therefore$ One root lies between 2 and 3

Consider $x_0 = 2$ *and* $x_1 = 3$

By Bisection method $x_2 = \frac{x_0 + x_1}{2} = 2.5$

Calculating $f(x_2) = f(2.5) = -3.375 < 0$

$\therefore$ The root lies between x_2 *and* x_1

The second approximation is $x_3 = \frac{1}{2}(x_1 + x_2) = \frac{2.5+3}{2} = 2.75$

Now $f(x_3) = f(2.75) = 0.7969 > 0$

$\therefore$ The root lies between x_2 *and* x_3

Thus the third approximation to the root is

$$x_4 = \frac{1}{2}(x_2 + x_3) = 2.625$$

Again $f(x_4) = f(2.625) = -1.421 < 0$

$\therefore$ The root lies between x_3 *and* x_4

Fourth approximation is$x_5 = \frac{1}{2}(x_3 + x_4) = \frac{1}{2}(2.75 + 2.625) = 2.6875$

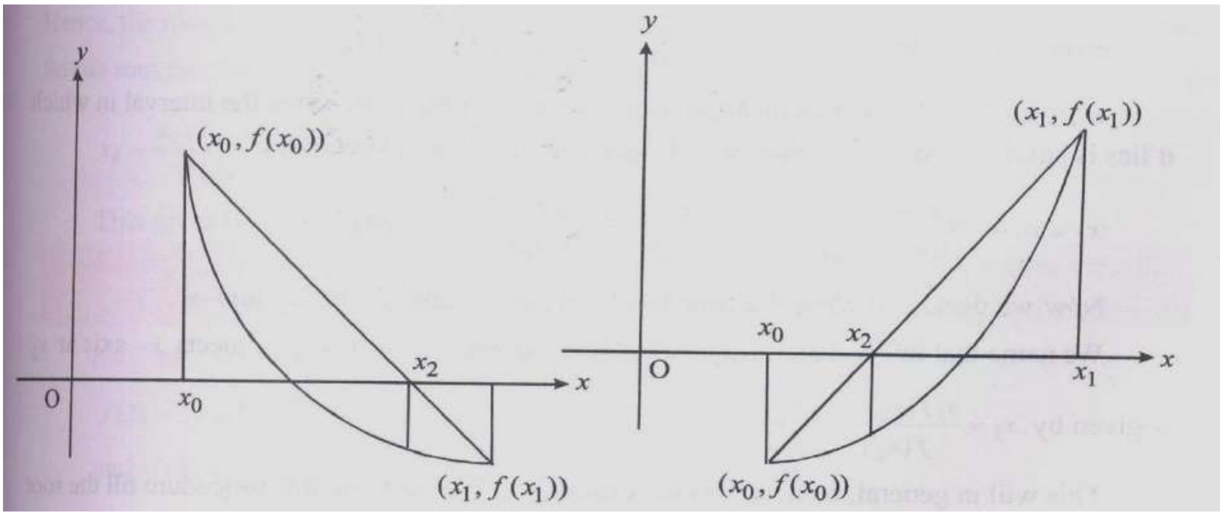
1.2. False Position Method (Regula – Falsi Method)

In the false position method we will find the root of the equation $f(x)=0$ Consider two initial approximate values x_0 and x_1 near the required root so that $f(x_0)$ and $f(x_1)$ have different signs. This implies that a root lies between x_0 and x_1. The curve $f(x)$ crosses x- axis only once at the Point x_2 lying between the points x_0 $and x_1$. Consider the point $A=(x_0, f(x_0))$ and $B=(x_1, f(x_1))$ on the graph and suppose they are connected by a straight line. Suppose this line cuts x-axis atx_2. We calculate the value of $f(x_2)$ at the point. If $f(x_0)$ and $f(x_2)$ are of opposite signs, then the root lies between x_0 and x_2 and value x_1 is replaced by x_2

Otherwise the root lies between x_2 and x_1 and the value of x_0 is replaced by x_2.Another line is drawn by connecting the newly obtained pair of values. Again the point here cuts the x-axis is a closer approximation to the root. This process is repeated as many times as required to obtain the desired accuracy. It can be observed that the points

$x_2, x_3, x_4, \ldots$obtained converge to the expected root of the equation $y = f(x)$

The below graph shows how to execute Regula Falsi Method



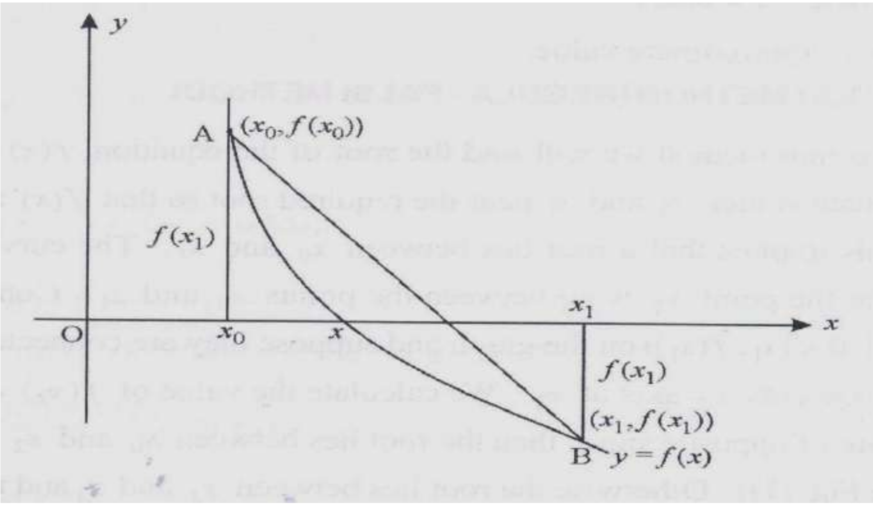
To Obtain the equation to find the next approximation to the root

Let $A=(x_0, f(x_0))$ and $B=(x_1, f(x_1))$ be the points on the curve $y=f(x)$ Then the equation to the chord AB is $\frac{y-f(x_0)}{x-x_0}=\frac{f(x_1)-f(x_0)}{x_1-x_0}------(1)$

At the point C where the line AB crosses the x – axis, where $f(x)=0$ $ie,$ $y=0$

From (1), we get $x=x_0-\frac{x_1-x_0}{f(x_1)-f(x_0)}f(x_0)-$ --------------(2)

x is given by (2) serves as an approximated value of the root, when the interval in which it lies is small. If the new value of x is taken as x_2 then (2) becomes



$$x_2=x_0-\frac{(x_1-x_0)}{f(x_1)-f(x_0)}f(x_0)$$
$$=\frac{x_0 f(x_1)-x_1 f(x_0)}{f(x_1)-f(x_0)}\rightarrow(3)$$ -------------(3)

Now we decide whether the root lies between

x_0 and x_2 $(or)\,x_2$ and x_1

We name that interval as (x_1, x_2) The line joining$(x_1, y_1),(x_2, y_2)$ meets x – axis at x_3 is given by $x_3=\frac{x_1 f(x_2)-x_2 f(x_1)}{f(x_2)-f(x_1)}$

This will in general, be nearest to the exact root. We continue this procedure till the root is found to the desired accuracy

The iteration process based on (3) is known as the method of false position

The successive intervals where the root lies, in the above procedure are named as

$(x_0, x_1), (x_1, x_2), (x_2, x_3)$ etc

Where $x_i < x_{i+1}$ and $f(x_0), f(x_{i+1})$ are of opposite signs.

Also $x_{i+1} = \dfrac{x_{i-1} f(x_i) - x_i f(x_{i-1})}{f(x_i) - f(x_{i-1})}$

PROBLEMS:

1. By using Regula - Falsi method, find an approximate root of the equation $x^4 - x - 10 = 0$ that lies between 1.8 and 2. Carry out three approximations

Sol. Let us take $f(x) = x^4 - x - 10$ and $x_0 = 1.8, x_1 = 2$

Then $f(x_0) = f(1.8) = -1.3 < 0$ and $f(x_1) = f(2) = 4 > 0$

Since $f(x_0)$ and $f(x_1)$ are of opposite signs, the equation $f(x) = 0$ has a root between x_0 *and* x_1

The first order approximation of this root is

$$x_2 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0)$$
$$= 1.8 - \frac{2 - 1.8}{4 + 1.3} \times (-1.3)$$
$$= 1.849$$

We find that $f(x_2) = -0.161$ so that $f(x_2)$ *and* $f(x_1)$ are of opposite signs. Hence the root lies between x_2 *and* x_1 and the second order approximation of the root is

$$x_3 = x_2 - \left[\frac{x_1 - x_2}{f(x_1) - f(x_2)}\right] . f(x_2)$$

$$= 1.8490 - \left[\frac{2 - 1.849}{0.159}\right] \times (-0.159)$$

$$= 1.8548$$

We find that $f(x_3) = f(1.8548)$

$$= -0.019$$

So that $f(x_3)$ *and* $f(x_2)$ are of the same sign. Hence, the root does not lie between x_2 *and* x_3.But $f(x_3)$ *and* $f(x_1)$ are of opposite signs. So the root lies between x_3 *and* x_1 and the third order approximate value of the root is $x_4 = x_3 - \left[\frac{x_1 - x_3}{f(x_1) - f(x_3)}\right] f(x_3)$

$$= 1.8548 - \frac{2 - 1.8548}{4 + 0.019} \times (-0.019)$$

$$= 1.8557$$

This gives the approximate value of x.

2. Find out the roots of the equation $x^3 - x - 4 = 0$ using False position method

Sol. Let $f(x) = x^3 - x - 4 = 0$

Then $f(0) = -4, f(1) = -4, f(2) = 2$

Since $f(1)$ *and* $f(2)$ have opposite signs the root lies between 1 and 2

By False position method $x_2 = \dfrac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)}$

$$x_2 = \frac{(1 \times 2) - 2(-4)}{2 - (-4)}$$

$$= \frac{2 + 8}{6} = \frac{10}{6} = 1.666$$

$$f(1.666) = (1.666)^3 - 1.666 - 4$$
$$= -1.042$$

Now, the root lies between 1.666 and 2

$$x_3 = \frac{1.666 \times 2 - 2 \times (-1.042)}{2 - (-1.042)} = 1.780$$
$$f(1.780) = (1.780)^3 - 1.780 - 4$$
$$= -0.1402$$

Now, the root lies between 1.780 and 2

$$x_4 = \frac{1.780 \times 2 - 2 \times (-0.1402)}{2 - (-0.1402)} = 1.794$$
$$f(1.794) = (1.794)^3 - 1.794 - 4$$
$$= -0.0201$$

Now, the root lies between 1.794 and 2

$$x_5 = \frac{1.794 \times 2 - 2 \times (-0.0201)}{2 - (-0.0201)} = 1.796$$
$$f(1.796) = (1.796)^3 - 1.796 - 4 = -0.0027$$

Now, the root lies between 1.796 and 2

$$x_6 = \frac{1.796 \times 2 - 2 \times (-0.0027)}{2 - (-0.0027)} = 1.796$$

The root is 1.796

1.3. Newton- Raphson Method:-

The Newton- Raphson method is a powerful and elegant method to find the root of an equation. This method is generally used to improve the results obtained by the previous methods.

Let x_0 be an approximate root of $f(x)=0$ and let $x_1 = x_0 + h$ be the correct root which implies that $f(x_1)=0$. We use Taylor's theorem and expand $f(x_1)= f(x_0+h)=0$

$$\Rightarrow f(x_0)+hf^1(x_0)=0$$

$$\Rightarrow h=-\frac{f(x_0)}{f^1(x_0)}$$

Substituting this in x_1 ,we get

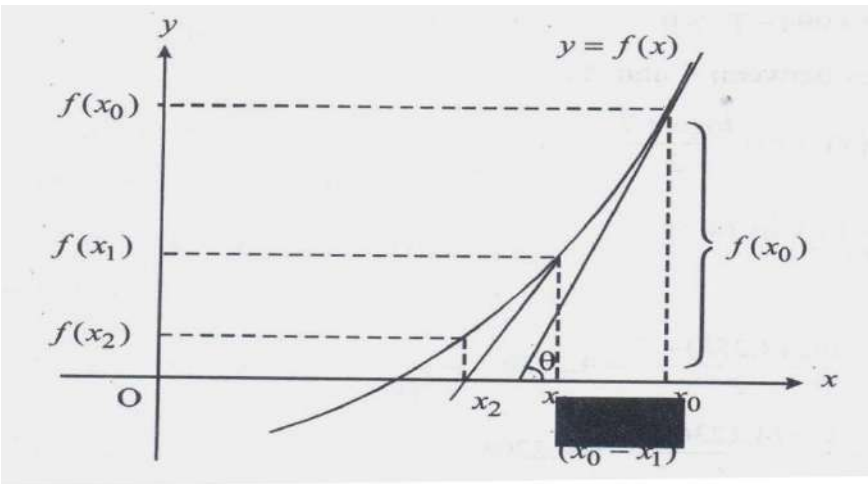
$$x_1 = x_0 + h$$

$$= x_0 - \frac{f(x_0)}{f^1(x_0)}$$

$\therefore x_1$ is a better approximation than x_0

Successive approximations are given by

$x_2, x_3 \ldots\ldots\ldots.. x_{n+1}$ Where $x_{i+1} = x_i - \frac{f(x_i)}{f^1(x_i)}$



PROBLEMS:

1. Apply Newton – Rapson method to find an approximate root, correct to three decimal places, of the equation $x^3-3x-5=0$, which lies near $x=2$

Sol:- Here $f(x)=x^3-3x-5=0 \;\; and \;\; f^1(x)=3(x^2-1)$

$\therefore$ The Newton – Raphson iterative formula

$$x_{i+1} = x_i - \frac{x_i^3 - 3x_i - 5}{3\left(x_i^2 - 1\right)} = \frac{2x_i^3 + 5}{3\left(x_i^2 - 1\right)}, i = 0,1,2....(1)$$

To find the root near $x = 2$, we take $x_0 = 2$ then (1) gives

$$x_1 = \frac{2x_0^3 + 5}{3\left(x_0^2 - 1\right)} = \frac{16 + 5}{3(4 - 1)} = \frac{21}{9} = 2.3333$$

$$x_2 = \frac{2x_1^3 + 5}{3\left(x_1^2 - 1\right)} = \frac{2 \times (2.3333)^3 + 5}{3\left[(2.3333)^2 - 1\right]} = 2.2806$$

$$x_3 = \frac{2x_2^3 + 5}{3(x_2^3 - 1)} = \frac{2 \times (2.2806)^3 + 5}{3[(2.2806)^2 - 1]} = 2.2790$$

$$x_4 = \frac{2 \times (2.2790)^3 + 5}{3[(2.2790)^2 - 1]} = 2.2790$$

Since x_3 and x_4 are identical up to 3 places of decimal, we take $x_4 = 2.279$ as the required root, correct to three places of the decimal

2. Using Newton – Raphson method

a) Find square root of a number

b) Find reciprocal of a number

Sol. a) **Square root:-**

Let $f(x) = x^2 - N = 0$, where N is the number whose square root is to be found. The solution to $f(x)$ is then $x = \sqrt{N}$

Here $f'(x) = 2x$

By Newton-Raphson technique

$$x_{i+1} = x_i - \frac{f(x_i)}{f^1(x_i)} = x_i - \frac{x_i^2 - N}{2x_i}$$

$$\Rightarrow x_{i+1} = \frac{1}{2}\left[x_i + \frac{N}{x_i}\right]$$

Using the above iteration formula the square root of any number N can be found to any desired accuracy. For example, we will find the square root of $N = 24$.

Let the initial approximation be $x_0 = 4.8$

$$x_1 = \frac{1}{2}\left(4.8 + \frac{24}{4.8}\right) = \frac{1}{2}\left(\frac{23.04 + 24}{4.8}\right) = \frac{47.04}{9.6} = 4.9$$

$$x_2 = \frac{1}{2}\left(4.9 + \frac{24}{4.9}\right) = \frac{1}{2}\left(\frac{24.01 + 24}{4.9}\right) = \frac{48.01}{9.8} = 4.898$$

$$x_3 = \frac{1}{2}\left(4.898 + \frac{24}{4.898}\right) = \frac{1}{2}\left(\frac{23.9904 + 24}{4.898}\right) = \frac{47.9904}{9.796} = 4.898$$

Since $x_2 = x_3$, therefore the solution to $f(x) = x^2 - 24 = 0$ is 4.898. That means,

The square root of 24 is 4.898

b) Reciprocal:-

Let $f(x) = \frac{1}{x} - N = 0$ where N is the number whose reciprocal is to be found

The solution to $f(x)$ is then $x = \frac{1}{N}$. Also, $f^1(x) = \frac{-1}{x^2}$

To find the solution for $f(x) = 0$, apply Newton – Raphson method

$$x_{i+1} = x_i - \frac{\left(\frac{1}{x_i} - N\right)}{-1/x_i^2} = x_i(2 - x_i N)$$

For example, the calculation of reciprocal of 22 is as follows

Assume the initial approximation be $x_0 = 0.045$

$$\therefore x_1 = 0.045(2 - 0.045 \times 22)$$
$$= 0.045(2 - 0.99)$$
$$= 0.0454(1.01) = 0.0454$$
$$x_2 = 0.0454(2 - 0.0454 \times 22)$$
$$= 0.0454(2 - 0.9988)$$
$$= 0.0454(1.0012) = 0.04545$$
$$x_3 = 0.04545(2 - 0.04545 \times 22)$$
$$= 0.04545(1.0001) = 0.04545$$

$$x_4 = 0.04545(2 - 0.04545 \times 22)$$
$$= 0.04545(2 - 0.99998)$$
$$= 0.04545(1.00002)$$
$$= 0.0454509$$

$\therefore$ The reciprocal of 22 is 0.04545

3. Find by Newton's method, the real root of the equation $xe^x - 2 = 0$ correct to three decimal places.

Sol. Let $f(x) = xe^x - 2 \rightarrow (1)$

Then $f(0) = -2$ and $f(1) = e - 2 = 0.7183$

So root of $f(x)$ lies between 0 and 1

It is near to 1. So we take $x_0 = 1$ and $f^1(x) = xe^x + e^x$ *and* $f^1(1) = e + e = 5.4366$

$\therefore$ By Newton's Rule

First approximation $x_1 = x_0 - \dfrac{f(x_0)}{f^1(x_0)}$

$$= 1 - \frac{0.7183}{5.4366} = 0.8679$$

$\therefore f(x_1) = 0.0672 \qquad f^1(x_1) = 4.4491$

The second approximation $x_2 = x_1 - \frac{f(x_1)}{f^1(x_1)}$

$$= 0.8679 - \frac{0.0672}{4.4491}$$
$$= 0.8528$$

$\therefore$ Required root is 0.853 correct to 3 decimal places.

1.4.GAUSS JORDAN METHOD:

SYSTEM OF NON HOMOGENEOUS LINEAR EQUATIONS

An equation of the form $a_1x_1 + a_2x_2 + ...+ a_nx_n = b$ where $x_1, x_2, ..., x_n$ are unknowns and $a_1, a_2, ..., a_n$, b are constants is called linear equation in n unknowns .

Definition: Consider the system of m linear equations in n unknowns $x_1, x_2, ..., x_n$ as given below:

$$a_{11}x_1 + a_{12}x_2 + ...+ a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + ...+ a_{2n}x_n = b_2$$

....

$$a_{i1}x_1 + a_{i2}x_2 + ...+ a_{in}x_n = b_i$$

....

$$a_{m1}x_1 + a_{m2}x_2 + ...+ a_{mn}x_n = b_m$$

The number a_{ij}'s are known as coefficient and $b_1, b_2, ...b_m$ are constants. An ordered n-tuple $(x_1, x_2, ..., x_n)$ satisfying all the equations simultaneously is called a solution of system.

Non-Homogeneous system:

If all $b_i \neq 0$ i.e.at least one $b_i \neq 0$.

Matrix Representation:

The above system of linear non Homogeneous equations can be written in Matrix form as AX=B

$$\begin{bmatrix} a_{11} & a_{12}\cdots\cdots\cdots & a_{1n} \\ a_{21} & a_{22}\cdots\cdots\cdots & a_{2n} \\ a_{m1} & a_{m2}\cdots\cdots\cdots & a_{mn} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_m \end{bmatrix}$$

Augmented Matrix:

It is denoted by [A/B] or [A B] is obtained by Augmenting A by the column B.

$$\therefore [A/B] = \begin{bmatrix} a_{11} & a_{12}\cdots\cdots\cdots & a_{1n} & b_1 \\ a_{21} & a_{22}\cdots\cdots\cdots & a_{2n} & b_2 \\ a_{m1} & a_{m2}\cdots\cdots\cdots & a_{mn} & b_m \end{bmatrix}$$

By reducing [A /B] into its row echelon form the existence and uniqueness of solution

AX = B exists.

NOTE:

Given a system, we do not know in general whether it has a solution or not .If there is at least one solution , then the system is said to be consistent .If does not have any solution then the system is inconsistent.

CONSISTENT**:** A system is said to be consistent if it has at least one solution

NOTE: Here rank is denoted by ρ

Gauss Jordan Method: In Gauss Jordan method augmented matrix [A/B] can be reduced to identity matrix and column matrix by elementary row operations. Finally last column gives solutions of given linear system.

The Augmented matrix [A/B] can be reduced as follows by elementary row operations

$$[A/B] = \begin{bmatrix} 1 & 1 & 2 & 4 \\ 2 & -1 & 3 & 9 \\ 3 & -1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0\alpha \\ 0 & 1 & 0\beta \\ 0 & 0 & 1\gamma \end{bmatrix}$$

Then last column is the solution set of given linear system

For Non Homogeneous System, The system AX = B is consistent i.e it has a solution.

The system is inconsistent i.e. it has no solution.

NOTE: Find the rank A and rank [A /B] by reducing the augmented matrix [A /B] to Echelon form by elementary row operations. Then the matrix A will be reduced to Echelon form.

This procedure is illustrated through the following examples.

Example 1: Find whether the following equations are consistent, if so solve them

By Gauss Jordan method x + y + 2z= 4; 2x – y + 3z = 9; 3x – y - z = 2.

Solution: The given equations can be written in the matrix form as $\begin{bmatrix} 1 & 1 & 2 \\ 2 & -1 & 3 \\ 3 & -1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 9 \\ 2 \end{bmatrix}$

i.e. .AX = B Use Gauss Jordan method

The Augmented matrix $[A/B] = \begin{bmatrix} 1 & 1 & 2 & 4 \\ 2 & -1 & 3 & 9 \\ 3 & -1 & -1 & 2 \end{bmatrix}$

Applying $R_2 \rightarrow R_2 - 2R_1$ and $R_3 \rightarrow R_3 - 3R_1$

$$[A/B] = \begin{bmatrix} 1 & 1 & 2 & 4 \\ 0 & -3 & -1 & 1 \\ 0 & -4 & -7 & -10 \end{bmatrix}$$

Applying $R_3 \rightarrow 3R3 - 4R_2$

$$[A / B] = \begin{bmatrix} 1 & 1 & 2 & 4 \\ 0 & -3 & -1 & 1 \\ 0 & 0 & -17 & -34 \end{bmatrix}$$

Since Rank of A = 3 & Rank of [A / B] = 3

Since the number of non- zero rows of matrix A is 3

Since the number of non- zero rows of matrix [A /B] is 3

∴ Rank of A = Rank of [A B]

i.e. $\rho(A) = \rho(AB)$

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & -3 & -1 \\ 0 & 0 & -17 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ -34 \end{bmatrix}$$

$\Rightarrow$ R3← R3/(-17)

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & -3 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$$

$\Rightarrow$ R1← R1-2 R3 and R2→R2+R3

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix}$$

Next perform R2/(-3) and R1→R1-R2

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

Then solution set X= $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$

$\therefore$ x = 1, y = -1 , z = 2 is the solution.

Example 2: Using Gauss Jordan method solve linear equations given below

x + 2y+ 2z = 2; 3x - 2y - z = 5: 2x - 5y + 3z = -4; x + 4y + 6z = 0.

Solution: The given equations can be written in the matrix form as AX = B

i.e. $\begin{bmatrix} 1 & 2 & 2 \\ 3 & -2 & -1 \\ 2 & -5 & 3 \\ 1 & 4 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ -4 \\ 0 \end{bmatrix}$

The Augmented matrix [A/ B] = $\begin{bmatrix} 1 & 2 & 2 & 2 \\ 3 & -2 & -1 & 5 \\ 2 & -5 & 3 & -4 \\ 1 & 4 & 6 & 0 \end{bmatrix}$

Use Gauss Jordan method

Applying $R_2 \to R_2 - 3R_1$; $R_3 \to R_3 - 2R_1$; $R_4 \to R_4 - R_1$

[A /B] = $\begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & -8 & -7 & -1 \\ 0 & -9 & -1 & -8 \\ 0 & 2 & 4 & -2 \end{bmatrix}$

Applying $R_3 \to 8R_3 - 9R_2$;; $R_4 \to 4R_4 + R_2$,we get

[A/B] = $\begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & -8 & -7 & -1 \\ 0 & 0 & 55 & -55 \\ 0 & 0 & 9 & -9 \end{bmatrix}$

Applying $R_3 \to R_3/55$; $R_4 \to R_4/9$

$$[A/B] \approx \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & -8 & -7 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

Applying $R_4 \to R_4 - R_3$

$$[A/B] \approx \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & -8 & -7 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since Rank of A = 3 & Rank of [A/ B] = 3

∴ Rank of A = Rank of [A /B]

i.e. $\rho(A) = \rho(AB)$

The given system is consistent, so it has a solution.

We have $\begin{bmatrix} 1 & 2 & 2 \\ 0 & -8 & -7 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ -1 \\ 0 \end{bmatrix}$

$\Rightarrow$ Apply R1-2R3,R2+7R3

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & -8 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ -8 \\ -1 \\ 0 \end{bmatrix}$$

Next R2/(-8) and R1-2R2 then

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -1 \\ 0 \end{bmatrix}$$

$\therefore x = 2$, $y = 1$, $z = -1$ is the solution.

1.5. Gauss Siedel Method:

Algorithm: Consider the linear system of equations as

(i) $a_1 x+b_1y+c_1z=d_1$

$a_2x+b_2y+c_2z=d_2$

$a_3x+b_3y+c_3z=d_3$

(ii) If a_1,b_2, c_3 are large as compared with other coefficients , then solve them for x,y,z respectively .

The system can be written in the below form

$$X = \frac{1}{a1} (d_1-b_1y-c_1z)$$

$$Y = \frac{1}{b2} (d_2-a_2x-c_2z)$$

$$Z= \frac{1}{c3} (d_3-a_3x-b_3y)$$

(iii) First iteration: We can calculate first iteration values in the following equations

$$X^1 = \frac{1}{a1} (d_1-b_1y^0+c_1z^0)$$

$$Y^1 = \frac{1}{b2} (d_2-a_2x^1-c_2z^0)$$

$$Z^1= \frac{1}{c3} (d_3-a_3x^1-b_3y^1)$$

(iv) Second iteration: Formulas for second iteration

$$X^2 = \frac{1}{a1} (d_1-b_1y^1+c_1z^1)$$

$$Y^2= \frac{1}{b2} (d_2-a_2x^2-c_2z^1)$$

$$Z^2= \frac{1}{c3} (d_3-a_3x^2-b_3y^2)$$

First take initial values zeroes as new approximation for an un known value found, it is immediately used in next step. We continued these processes up to two successive iterations are approximately equal. This procedure is called as Gauss Siedal iteration method.

PROBLEMS:

1. Solve by Gauss Siedal method 10x+y+z=12 , 2x+10y+z=13 , 2x+2y+10z=14

Sol. Given equations are 10x+y+z=12----(1)

2x+10y+z=13----(2)

2x+2y+10z=14---(3)

From (1) $x=\frac{1}{10}(12-y-z)$

From (2) $y=\frac{1}{10}(13-2x-z)$

From (3) $z=\frac{1}{10}(14-2x-2y)$

First iteration: $X^1=\frac{1}{10}(12-y^0-z^0)=\frac{1}{10}(12-0-0)=1.2$

$Y^1=\frac{1}{10}(13-2x^1-z^0)=\frac{1}{10}(13-2(1.2)-0)=1.06$

$Z^1=\frac{1}{10}(14-2x^1-2y^1)=\frac{1}{10}(14-2(1.2)-2(1.06))=0.948$

Second iteration: $X^2=\frac{1}{10}(12-y^1-z^1)=\frac{1}{10}(12-1.2-1.06)=0.999$

$Y^2=\frac{1}{10}(13-2x^2-z^1)=\frac{1}{10}(13-2(0.999)-0.948)=1.005$

$Z^2=\frac{1}{10}(14-2x^2-2y^2)=\frac{1}{10}(14-2(0.999)-2(1.005))=0.999$

Third iteration: $X^3=\frac{1}{10}(12-y^2-z^2)=\frac{1}{10}(12-1.005-0.999)=1$

$Y^3=\frac{1}{10}(13-2x^3-z^2)=\frac{1}{10}(13-2(1)-0.999)=1$

$Z^3=\frac{1}{10}(14-2x^3-2y^3)=\frac{1}{10}(14-2(1)-2(1))=1$

Fourth iteration: $X^4=\frac{1}{10}(12-y^3-z^3)=\frac{1}{10}(12-1-1)=1$

$Y^4=\frac{1}{10}(13-2x^4-z^3)=\frac{1}{10}(13-2(1)-1)=1$

$Z^4=\frac{1}{10}(14-2x^4-2y^4)=\frac{1}{10}(14-2(1)-2(1))=1$

Since third and fourth iterations are equal then desired set of solutions are

X=1, y=1 , z=1

2. Using Gauss Siedal method solve the linear system 20x+y-2z=17,3x+20y-z=-18,2x-3y+20z=25
3. Solve 6x+y+z=105,4x+8y+3z=155,5x+4y-10z=65 by Gauss Siedal method.

9. Practice Quiz

1. Newton's iterative formula for finding the Cube root of a number N is x_{n+1} = [b]

a) $\frac{1}{3}\left[2x_n + \frac{N}{x_n{}^2}\right]$

b) $\frac{1}{3}\left[2x_n + \frac{N}{x_n{}^3}\right]$

c) $\frac{1}{3}\left[2x_n - \frac{N}{x_n{}^2}\right]$

d) $\frac{1}{3}\left[2x_n - \frac{N}{x_n{}^3}\right]$

2. Iteration formula in Newton-Raphson method is [b]

a) $x_{n+1} = x_n + \frac{f(x_n)}{f^1(x_n)}$

b) $x_{n+1} = x_n - \frac{f(x_n)}{f^1(x_n)}$

c) $x_{n+1} = x_n + \frac{f^1(x_n)}{f(x_n)}$

d) $x_{n+1} = x_n - \frac{f^1(x_n)}{f(x_n)}$

3. Which of the following is an algebraic equation ……….. [b]

a) $x^2 - \log x - 1.2 = 0$

b) $x^3 + 2x^2 + x + 1 = 0$

c) $\cos x = xe^x$

d) $xe^x - 1 = 0$

4. Which of the following is a transcendental equation… [a]

a) $x^2 - \log x = 1.2$

b) $x^3 + 2x^2 + x + 1 = 0$

c) $x^3 - 3x - 5 = 0$

d) $x^3 - 5x + 1 = 0$

5. Using the false position method, the formula for the approximate root of the equation $f(x) = 0$ is………… [a]

a) $x = \dfrac{af(b) - bf(a)}{f(b) - f(a)}$

b) $x = \dfrac{bf(b) - af(a)}{f(b) - f(a)}$

c) $x = \dfrac{af(b) + bf(a)}{f(b) + f(a)}$

d) $x = \dfrac{bf(b) + af(a)}{f(b) + f(a)}$

6. If the root of the equation $x^3 - 6x + 4 = 0$ lies between 0 & 1, then the first approximation of the required root using Newton-Raphson method is……. [c]

a)0.55555
b)0.4444
c)0.77777
d)0.66666

7. The n^{th} order difference of a polynomial of n^{th} degree is______ [a]

a) Constant

8. LECTURE NOTES

Interpolation

Introduction:-

If we consider the statement $y = f(x)x_0 \le x \le x_n$ we understand that we can find the value of y, corresponding to every value of x in the range $x_0 \le x \le x_n$. If the function $f(x)$ is single valued and continuous and is known explicitly then the values of $f(x)$ for certain values of x like $x_0, x_1, ...x_n$ can be calculated. The problem now is if we are given the set of tabular values

$$x : x_0 \quad x_1 \quad x_2........x_n$$
$$y : y_0 \quad y_1 \quad y_2........y_n$$

Satisfying the relation $y = f(x)$ and the explicit definition of $f(x)$ is not known, then it is possible to find a simple function say $f(x)$ such that $f(x)$ and $\phi(x)$ agree at the set of tabulated points. This process to finding $\phi(x)$ is called interpolation. If $\phi(x)$ is a polynomial then the process is called polynomial interpolation and $\phi(x)$ is called interpolating polynomial. In our study we are concerned with polynomial interpolation

Errors in Polynomial Interpolation:-

Suppose the function $y(x)$ which is defined at the points $(x_i, y_i) i = 0,1,2,3----n$ is continuous and differentiable $(n+1)$ times let $\phi_n(x)$ be polynomial of degree not exceeding n such that $\phi_n(x_i) = y_i, i = 1,2---n \rightarrow (1)$ be the approximation of $y(x)$ using this $\phi_n(x_i)$ for other value of x, not defined by (1) the error is to be determined

Since $y(x) - \phi_n(x) = 0$ for $x - x_0, x_1, ...x_n$ we put

$$y(x) - \phi_n(x) = L\pi_{n+1}(x)$$

Where $\pi_{n+1}(x)=(x-x_0).........(x-x_n)\rightarrow(3)$ and L to be determined such that the equation (2) holds for any intermediate value of x such as $x=x^1, x_0<x^1<x_n$

Clearly $L=\dfrac{y(x^1)-\phi_n(x^1)}{\pi_{n+1}(x^1)}\rightarrow(4)$

We construct a function $F(x)$ such that $F(x)=F(x_n)=F(x^1)$. Then $F(x)$ vanishes $(n+2)$ times in the interval $[x_0,x_n]$. Then by repeated application of Rolle's theorem. $F^1(x)$ Must be zero $(n+1)$ times, $F^{11}(x)$ must be zero n times……. in the interval $[x_0,x_n]$. Also $F^{n+1}(x)=0$ once in this interval. Suppose this point is $x=\varepsilon$, $x_0<\varepsilon<x_n$ differentiate (5) $(n+1)$ times with respect to x and putting $x=\varepsilon$, we get

$y^{n+1}(\varepsilon)-L(n+1)!=0$ Which implies that $L=\dfrac{y^{n+1}(\varepsilon)}{(n+1)!}$

Comparing (4) and (6) , we get

$$y(x^1)-\phi_n(x^1)=\frac{y^{n+1}(\varepsilon)}{(n+1)!}\pi_{n+1}(x^1)$$

This can be written as $y(x)-\phi_n(x)=\dfrac{\pi_{n+1}(x)}{(n+1)!}y^{n+1}(\varepsilon)$

This given the required expression $x_0<\varepsilon<x_n$ for error

2.1. Finite Differences:-

1. **Introduction:-**

In this chapter, we introduce what are called the forward, backward and central differences of a function $y=f(x)$. These differences and three standard examples of finite differences and play a fundamental role in the study of differential calculus, which is an essential part of numerical applied mathematics

2. **Forward Differences:-**

Consider a function $y = f(x)$ of an independent variable x. let $y_0, y_1, y_2, \ldots. y_r$ be the values of y corresponding to the values $x_0, x_1, x_2 \ldots. x_r$ of x respectively. Then the differences $y_1 - y_0, y_2 - y_1 -----$ are called the first forward differences of y, and we denote them by $\Delta y_0, \Delta y_1, \ldots\ldots.$ that is

$$\Delta y_0 = y_1 - y_0, \Delta y_1 = y_2 - y_1, \Delta y_2 = y_3 - y_2 \ldots\ldots\ldots$$

In general $\Delta y_r = y_{r+1} - y_r \therefore r = 0,1,2 -----$

Here, the symbol Δ is called the forward difference operator

The first forward differences of the first forward differences are called second forward differences and are denoted by $\Delta^2 y_0, \Delta^2 y_1 \ldots\ldots$ that is

$$\Delta^2 y_0 = \Delta y_1 - \Delta y_0$$
$$\Delta^2 y_1 = \Delta y_2 - \Delta y_1$$

In general $\Delta^2 y_r = \Delta y_{r+1} - \Delta y_r \quad r = 0,1,2 \ldots\ldots.$ similarly, the n[th] forward differences are defined by the formula.

$$\Delta^n y_r = \Delta^{n-1} y_{r+1} - \Delta^{n-1} y_r \quad r = 0,1,2 \ldots\ldots.$$

While using this formula for $n = 1$, use the notation $\Delta^0 y_r = y_r$ and we have $\Delta^n y_r = 0 \forall n = 1,2 \ldots\ldots \text{ and } r = 0,2, \ldots\ldots\ldots$ the symbol Δ^n is referred as the n[th] forward difference operator.

3. Forward Difference Table:-

The forward differences are usually arranged in tabular columns as shown in the following table called a forward difference table

Values of x	Values of y	First differences	Second differences	Third differences	Fourth differences
x_o	y_0				
		$\Delta y_0 = y_1 - y_0$			
x_1	y_1		$\Delta^2 y_0 = \Delta y_1 - y_0$		
		$\Delta y_1 = y_2 - y_1$		$\Delta^3 y_0 = \Delta^2 y_1 - \Delta^2 y_0$	
x_2	y_2		$\Delta^2 y_1 = \Delta y_2 - \Delta y_1$		$\Delta^4 y_0 = \Delta^3 y_1 - \Delta^3 y_0$
		$\Delta y_2 = y_3 - y_2$		$\Delta^3 y_1 = \Delta^2 y_2 - \Delta^2 y_1$	
x_3	y_3		$\Delta^2 y_2 = \Delta y_3 - \Delta y_2$		
X4	y_4	$= y_4 - y_3$			

Example -finite forward difference table for $y = x^3$

x	$y = f(x)$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1	1				
		7			
2	8		12		
		19		6	
3	27		18		0
		37		6	
4	64		24		0
		61		6	
5	125		30		
		91			
6	216				

4. Backward Differences:- As mentioned earlier, let $y_0, y_1 ...... y_r ......$ be the values of a function $y = f(x)$ corresponding to the values $x_0, x_1, x_2 .... x_r ......$ of x respectively. Then, $\nabla y_1 = y_1 - y_0, \nabla y_2 = y_2 - y_1, \nabla y_3 = y_3 - y_2, ...$ are called the first backward differences

In general $\nabla y_r = y_r - y_{r-1}$, $r = 1,2,3 \ldots\ldots\ldots \rightarrow (1)$

The symbol ∇ is called the backward difference operator, like the operator Δ, this operator is also a linear operator

Comparing expression (1) above with the expression (1) of section we immediately note that $\nabla y_r = \nabla y_{r-1}, r = 0,1,2 \ldots\ldots\ldots \rightarrow (2)$

The first backward differences of the first background differences are called second differences and are denoted by $\nabla^2 y_2, \nabla^2 y_3 - - - \nabla^2{}_r - - - -$

i.e.,.. $\nabla^2 y_2 = \nabla y_2 - \nabla y_1, \nabla^2 y_3 = \nabla y_3 - \nabla y_2 \ldots\ldots\ldots$

In general $\nabla^2 y_r = \nabla y_r - \nabla y_{r-1}, r = 2,3 \ldots\ldots \rightarrow (3)$ similarly, the n[th] backward differences are defined by the formula $\nabla^n y_r = \nabla^{n-1} y_r - \nabla^{n-1} y_{r-1}, r = n, n+1 \ldots\ldots \rightarrow (4)$ While using this formula, for n = 1 we employ the notation $\nabla^0 y_r = y_r$

If $y = f(x)$ is a constant function, then y = c is a constant, for all x, and we get $\nabla^n y_r = 0 \forall n$ the symbol ∇^n is referred to as the n[th] backward difference operator

5. Backward Difference Table:-

X	Y	∇y	$\nabla^2 y$	$\nabla^3 y$
x_0	y_0			
		∇y_1 =y1-y0		
x_1	y_1		$\nabla^2 y_2$	
		∇y_2 **=y2-y1**		$\nabla^3 y3$
x_2	y_2		$\nabla^2 y_3$	
		∇y_3 =y3-y2		
x_3	y_3			

6. Central Differences:-

With $y_0, y_1, y_2 .... y_r$ as the values of a function $y = f(x)$ corresponding to the values $x_1, x_2 ..... x_r .... \ of \ x$, we define the first central differences

$\delta y_{1/2}, \delta y_{3/2}, \delta y_{5/2} ----$ as follows

$$\delta y_{1/2} = y_1 - y_0, \delta y_{3/2} = y_2 - y_1, \delta y_{5/2} = y_3 - y_2 ----$$

$$\delta y_{r-1/2} = y_r - y_{r-1} \rightarrow (1)$$

The symbol δ is called the central differences operator. This operator is a linear operator comparing expressions (1) above with expressions earlier used on forward and backward differences we get

$$\delta y_{1/2} = \Delta y_0 = \nabla y_1, \delta y_{3/2} = \Delta y_1 = \nabla y_2 .....$$

In general $\delta y_{n+1/2} = \Delta y_n = \nabla y_{n+1}, n = 0,1,2...... \rightarrow (2)$

The first central differences of the first central differences are called the second central differences and are denoted by $\delta^2 y_1, \delta^2 y_2 ...$

Thus $\delta^2 y_1 = \delta_{3/2} - \delta y_{1/2}, \delta^2 y_2 = \delta_{5/2} - \delta_{3/2} ....$

$$\delta^2 y_n = \delta y_{n+1/2} - \delta y_{n-1/2} \rightarrow (3)$$

Higher order central differences are similarly defined. In general the n[th] central differences are given by

i) for odd $n : \delta^n y_{r-1/2} = \delta^{n-1} y_r - \delta^{n-1} y_{r-1}, r = 1,2.... \rightarrow (4)$

ii) for even $n : \delta^n y_r = \delta^{n-1} y_{r+1/2} - \delta^{n-1} y_{r-1/2}, r = 1,2.... \rightarrow (5)$

while employing for formula (4) for $n = 1$, we use the notation $\delta^0 y_r = y_r$

If y is a constant function, that is if $y = c$ a constant, then $\delta^n y_r = 0 \quad for \ \ all \ \ n \geq 1$

7. Central Difference Table

x_0	y_0	δy	$\delta^2 y$	$\delta^3 y$	$\delta^4 y$
		$\delta y_{1/2}$			
x_1	y_1		$\delta^2 y_1$		
		$\delta y_{2/2}$		$\delta^3 y_{3/2}$	
x_2	y_2		$\delta^2 y_2$		$\delta^4 y_2$
		$\delta y_{5/2}$		$\delta^3 y_{5/2}$	
x_3	y_3		$\delta^2 y_3$		
		$\delta y_{7/2}$			
x_4	y_4				

Example: Given $f(-2)=12, f(-1)=16, f(0)=15, f(1)=18, f(2)=20$ from the central difference table and write down the values of $\delta y_{3/2}, \delta^2 y_0$ *and* $\delta^3 y_{7/2}$ by taking $x_0 = 0$

Sol. The central difference table is

x	$y = f(x)$	δy	$\delta^2 y$	$\delta^3 y$	$\delta^4 y$
-2	12				
		4			
-1	16		-5		
		-1		9	
0	15		4		-14
		3		-5	
1	18		-1		

		2			
2	20				

5. Symbolic Relations and Separation of symbols:

We will define more operators and symbols in addition to Δ, ∇ and δ already defined and establish difference formulae by symbolic methods

Averaging Operator-Definition:- The averaging operator μ is defined by the equation

$$\mu y_r = \frac{1}{2}\left[y_{r+1/2} + y_{r-1/2}\right]$$

Shift Operator-Definition:- The shift operator E is defined by the equation $Ey_r = y_{r+1}$. This shows that the effect of E is to shift the functional value y_r to the next higher value y_{r+1}. A second operation with E gives $E^2 y_r = E(Ey_r) = E(y_{r+1}) = y_{r+2}$

Generalizing $E^n y^r = y_{r+n}$

Relationship Between Δ and E

We have
$$\Delta y_0 = y_1 - y_0$$
$$= Ey_0 - y_0 = (E-1)y_0$$
$$\Rightarrow \Delta = E - y\,(or)\,E = 1 + \Delta$$

$$\Delta y_0 = y_1 - y_0$$
$$= Ey_0 - y_0 = (E-1)y_0$$
$$\Rightarrow \Delta = E - y\,(or)\,E = 1 + \Delta$$

Some more relations

$$\Delta^3 y_0 = (E-1)^3 y_0 = \left(E^3 - 3E^2 + 3E - 1\right)y_0$$
$$= y_3 - 3y_2 + 3y_1 - y_0$$

Inverse Shift Operator-Definition

Inverse operator E^{-1} is defined as $E^{-1}y_r = y_{r-1}$

In general $E^{-n}y_n = y_{r-n}$

We can easily establish the following relations

i) $\nabla \equiv 1 - E^{-1}$

ii) $\delta \equiv E^{1/2} - E^{-1/2}$

iii) $\mu = \frac{1}{2}\left(E^{1/2} + E^{-1/2}\right)$

iv) $\Delta = \nabla E = E^{1/2}$

v) $\mu^2 \equiv 1 + \frac{1}{4}\delta^2$

Differential Operator-Definition The operator D is defined as $Dy(x) = \frac{\partial}{\partial x}\left[y(x)\right]$

Relation between the Operators D and E

Using Taylor's series we have, $y(x+h) = y(x) + hy^1(x) + \frac{h^2}{2!}y^{11}(x) + \frac{h^3}{3!}y^{111}(x) + ----$

This can be written in symbolic form

$$Ey_x = \left[1 + hD + \frac{h^2D^2}{2!} + \frac{h^3D^3}{3!} + ----\right]y_x = e^{hD}.y_x$$

We obtain in the relation $E = e^{hD} \rightarrow (3)$

- If $f(x)$ is a polynomial of degree n and the values of x are equally spaced then $\Delta^n f(x)$ is constant

Proof:

Let $f(x) = a_0x^n + a_1x^{n-1} + ------ + a_{n-1}x + a_n$ where $a_0, a_1, a_2....a_n$ are constants and $a_0 \neq 0$. If h is the step- length, we know the formula for the first forward difference

$$\Delta f(x) = f(x+h) - f(x) = \left[a_0(x+h)^n + a_1(x+h)^{n-1} + ----- + a_{n-1}(x+h) + a_n\right]$$
$$-\left[a_0x^n + a_1x^{n-1} + ----- + a_{n-1}x + a_n\right]$$

$$= a_0\left[\left\{x^n + n.x^{n-1}h + \frac{n(n-1)}{2!}x^{n-2}.h^2 + ---\right\} - x^n\right] +$$

$$a_1\left[\left\{x^{n-1} + (n-1)x^{n-2}.h + \frac{(n-1)(n-2)}{2!}x^{n-3}.h^2 + ---\right\} - x^{n-1}\right] +$$

$$------ + a_{n-1}h$$

$$= a_0nhx^{n-1} + b_2x^{n-2} + b_3x^{n-3} + ----- + b_{n-3}x + b_{n-2}$$

Where $b_2, b_3, ...b_{n-2}$ are constants. Here this polynomial is of degree $(n-1)$, thus, the first difference of a polynomial of nth degree is a polynomial of degree $(n-1)$

Now

$$\Delta^2 f(x) = \Delta[\Delta f(x)]$$

$$= \Delta\left[a_0nh.x^{n-1} + b_2x^{n-2} + b_3x^{n-3} + ----- + b_{n-1}x + b_{n-2}\right]$$

$$= a_0nh\left[(x+h)^{n-1} - x^{n-1}\right] + b_2\left[(x+h)^{n-2} - x^{n-2}\right] + ---- + b_{n-1}\left[(x+h) - x\right]$$

$$= a_0n^{(n-1)}h^2x^{n-2} + c_3x^{n-3} + ------ + c_{n-4}x + c_{n-3}$$

Where $c_3....c_{n-3}$ are constants. This polynomial is of degree $(n-2)$

Thus, the second difference of a polynomial of degree n is a polynomial of degree $(n-2)$ continuing like this we get $\Delta^n f(x) = a_0n(n-1)(n-2)-----2.1.h^n = a_0h^n(n!)$

$\therefore$ which is constant

Note:-

1. As $\Delta^n f(x)$ is a constant, it follows that $\Delta^{n+1} f(x) = 0, \Delta^{n+2} f(x) = 0, ...$
2. The converse of above result is also true that is, if $\Delta^n f(x)$ is tabulated at equal spaced intervals and is a constant, then the function $f(x)$ is a polynomial of degree n

Example:-

1. Form the forward difference table and write down the values of $\Delta f(10)$, $\Delta^2 f(10), \Delta^3 f(15)$ *and* $\Delta^4 y(15)$

x	10	15	20	25	30	35
y	19.97	21.51	22.47	23.52	24.65	25.89

Sol.

x	Y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
10	**19.97f(10)**					
		1.54$\Delta f(10)$				
15	**21.51f(15)**		**-0.58**$\Delta^2 f(10)$			
		0.96		0.67		
20	**22.47f(20)**		0.09		- 0.68	
		1.05		**- 0.01**$\Delta^3 f(15)$		0.72
25	**23.52f(25)**		0.08		**0.04** $\Delta^4 f(15)$	
		1.13		0.03		
30	**24.65f(30)**		0.11			
		1.24				
35	**25.89f(35)**					

We note that the values of x are equally spaced with step- length h = 5

Note: - $\therefore x_0 = 10, x_1 = 15 ----x_5 = 35$ and

$y_0 = f(x_0) = 19.97$

$y_1 = f(x_1) = 21.51$

$y_5 = f(x_5) = 25.89$

$y_5 = f(x_5) = 25.89$

From table

$$\Delta f(10) = \Delta y_0 = 1.54$$
$$\Delta^2 f(10) = \Delta^2 y_0 = -0.58$$
$$\Delta^3 f(15) = \Delta^3 y_1 = -0.01$$
$$\Delta^4 f(15) = \Delta^3 y_1 = 0.04$$

2. **Evaluate**

$(i)\Delta \cos x$

$(ii)\Delta^2 \sin(px+q)$

$(iii)\Delta^n e^{ax+b}$

Sol. Let h be the interval of differencing

$$(i)\Delta \cos x = \cos(x+h) - \cos x$$
$$= -2\sin\left(x+\frac{h}{2}\right)\sin\frac{\text{h}}{2}$$

$(ii)\Delta \sin(px+q) = \sin\left[p(x+h)+q\right] - \sin(px+q)$ $\Delta f(x) = f(x+h) - f(x)$,for ward formula

$$= 2\cos\left(px+q+\frac{ph}{2}\right)\sin\frac{ph}{2}$$
$$= 2\sin\frac{ph}{2}\sin\left(\frac{\pi}{2}+px+q+\frac{ph}{2}\right)$$

$\nabla f(x) = f(x) - f(x-h)$ back ward formula

$$\Delta^2 \sin(px+q) = 2\sin\frac{ph}{2}\Delta\left[\sin(px+q)+\frac{1}{2}(\pi+ph)\right]$$

$$= \left[2\sin\frac{ph}{2}\right]^2 \sin\left[px+q+\frac{1}{2}(\pi+ph)\right]$$

$$(iii)\ \Delta e^{ax+b} = e^{a(x+h)+b} - e^{ax+b}$$
$$= e^{(ax+b)}\left(e^{ah-1}\right)$$
$$\Delta^2 e^{ax+b} = \Delta\left[\Delta\left(e^{ax+b}\right)\right] - \Delta\left[\left(e^{ah}-1\right)\left(e^{ax+b}\right)\right]$$
$$= \left(e^{ah}-1\right)^2 \Delta\left(e^{ax+h}\right)$$
$$= \left(e^{ah}-1\right)^2 e^{ax+b}$$

Proceeding on, we get $\Delta^n\left(e^{ax+b}\right) = \left(e^{ah}-1\right)^n e^{ax+b}$

3. Using the method of separation of symbols show that

$$\Delta^n \mu_{x-n} = \mu_{x-n} - n\mu_{x-1} + \frac{n(n-1)}{2}\mu_{x-2} + - - - - + (-1)^n \mu_{x-n}$$

Sol. To prove this result, we start with the right hand side. Thus

$$\mu x - n\mu x - 1 + \frac{n(n-1)}{2}\mu x - 2 + - - - - - - + (-1)^n \mu x - n$$
$$= \mu x - nE^{-1}\mu x + \frac{n(n-1)}{2}E^{-2}\mu x + - - - - - - + (-1)^n E^{-n}\mu x$$
$$= \left[1 - nE^{-1} + \frac{n(n-1)}{2}E^{-2} + - - - - - - + (-1)^n E^{-n}\right]\mu x = \left(1 - E^{-1}\right)^n \mu x$$
$$= \left(1 - \frac{1}{E}\right)^n \mu n = \frac{(E-1)^n}{E}\mu n$$
$$= \frac{\Delta^n}{E^n}\mu x = \Delta^n E^{-n}\mu x$$

$= \Delta^n \mu_{x-n}$ This is left hand side

4. Find the missing term in the following data

X	0	1	2	3	4
Y	1	3	9	-	81

Why this value is not equal to 3^3. Explain

Sol. Consider $\Delta^4 y_0 = 0$

x	Y	Δ	Δ2	Δ3	Δ4	
0	1	2	4	x-19	124-4x=0	
1	3	6	x-15	105-3x		
2	9	X-9	90-2x			
3	**X**	81-x				
4	81					

$\Rightarrow 4y_0 - 4y_3 + 5y_2 - 4y_1 + y_0 = 0$ 124-4x=0— **x=124/4=31**

Substitute given values we get

$81 - 4y_3 + 54 - 12 + 1 = 0 \Rightarrow y_3 = 31$

From the given data we can conclude that the given function is $y = 3^x$. To find y_3, we have to assume that y is a polynomial function, which is not so. Thus we are not getting $y = 3^3 = 27$

2.2. a. Newton's Forward Interpolation Formula:-

Let $y = f(x)$ be a polynomial of degree n and taken in the following form

$$y = f(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1) + b_3(x - x_0)(x - x_1)(x - x_2) + --- \\ + b_n(x - x_0)(x - x_1) ---- (x - x_{n-1}) \rightarrow (1)$$

This polynomial passes through all the points $[xi; yi]$ for i = 0 to n. there fore, we can obtain the y_i's by substituting the corresponding x_i's as

$$at \ \ x = x_0, y_0 = b_0$$
$$at \ \ x = x_1, y_1 = b_0 + b_1(x_1 - x_0)$$
$$at \ \ x = x_2, y_2 = b_0 + b_1(x_2 - x_0) + b_2(x_2 - x_0)(x_2 - x_1) \rightarrow (1)$$

Let 'h' be the length of interval such that x_i's represent

$$x_0, x_0+h, x_0+2h, x_0+3h----x_0+xh$$

This implies $x_1-x_0=h, x_2-x_0-2h, x_3-x_0=3h----x_n-x_0=nh \rightarrow (2)$

From (1) and (2), we get

$$y_0=b_0$$
$$y_1=b_0+b_1h$$
$$y_2=b_0+b_12h+b_2(2h)h$$
$$y_3=b_0+b_13h+b_2(3h)(2h)+b_3(3h)(2h)h$$

....

....

$$y_n=b_0+b_1(nh)+b_2(nh)(n-1)h+----+b_n(nh)[(n-1)h][(n-2)h]\rightarrow(3)$$

Solving the above equations for $b_0, b_{11}, b_2.....b_n$, we get $b_0=y_0$

$$b_1=\frac{y_1-b_0}{h}=\frac{y_1-y_0}{h}=\frac{\Delta y_0}{h}$$
$$b_2=\frac{y_2-b_0-b_12h}{2h^2}=y_2-y_0-\frac{(y_1-y_0)}{h}2h$$

$$=\frac{y_2-y_0-2y_1-2y_0}{2h^2}=\frac{y_2-2y_1+y_0}{2h^2}=\frac{\Delta^2y_0}{2h^2}$$
$$\therefore b_2=\frac{\Delta^2y_0}{2!h^2}$$

Similarly, we can see that

$$b_3=\frac{\Delta^3y_0}{3!h^3}, b_4=\frac{\Delta^4y_0}{4!h^4}------b_n=\frac{\Delta^ny_0}{n!h^n}$$
$$\therefore y=f(x)=y_0+\frac{\Delta y_0}{h}(x-x_0)+\frac{\Delta^2y_0}{2!h^2}(x-x_0)(x-x_1)$$

$$+\frac{\Delta^3y_0}{3!h^3}(x-x_0)(x-x_1)(x-x_2)+----+$$
$$+\frac{\Delta^ny_0}{n!h^n}(x-x_0)(x-x_1)---(x-x_{n-1})\rightarrow(3)$$

If we use the relationship $x=x_0+ph \Rightarrow x-x_0=ph,\ where\ p=0,1,2,.....n$

Then

$$x-x_1=x-(x_0+h)=(x-x_0)-h$$
$$=ph-h=(p-1)h$$
$$x-x_2=x-(x_1+h)=(x-x_1)-h$$
$$=(p-1)h-h=(p-2)h$$

....

$$x-x_i=(p-i)h$$

....

$$x-x_{n-1}=\left[p-(n-1)\right]h$$

Equation (3) becomes for $p=\frac{x-x_0}{h}$

$$y=f(x)=f(x_0+ph)=y_0+p\Delta y_0+\frac{p(p-1)}{2!}\Delta^2 y_0+\frac{p(p-1)(p-2)}{3!}\Delta^3 y_0+-----+$$
$$\frac{p(p-1)(p-2)----(p-(n-1))}{n!}\Delta^n y_0\to(4)$$

2.2. b. Newton's Backward Interpolation Formula:-

If we consider

$$y_n(x)=a_0+a_1(x-x_n)+a_2(x-x_n)(x-x_{n-1})+a_3(x-x_n)(x-x_{n-1})(x-x_{n-2})+-----(x-x_i)$$

and impose the condition that y and $y_n(x)$ should agree at the tabulated

points $x_n, x_n-1, ...x_2, x_1, x_0$

We obtain

$$y_n(x)=y_n+p\nabla y_n+\frac{p(p+1)}{2i}\nabla^2 y_n+---$$
$$\frac{p(p+1)----\left[p+(n-1)\right]}{n!}\nabla^n y_n+----\to(6)$$

Where $p=\frac{x-x_n}{h}$

This uses tabular values of the left of y_n . Thus this formula is useful formula is useful for interpolation near the end of the table values

Formula for Error in Polynomial Interpolation:-

If $y = f(x)$ is the exact curve and $y = \phi_n(x)$ is the interpolating curve, then the error in polynomial interpolation is given by

$$Error = f(x) - \phi_n(x) = \frac{(x-x_0)(x-x_1)----(x-x_n)}{(n+1)!} f^{n+1}(\varepsilon) \to (7)$$

for any x, where $x_0 < x < x_n$ and $x_0 < \varepsilon < x_n$

The error in Newton's forward interpolation formula is given by

$$f(x) - \phi_n(x) = \frac{p(p-1)(p-2).......(p-n)}{(n+1)!} \Delta^{n+1} f(\varepsilon)$$

Where $p = \frac{x - x_0}{h}$

The error in Newton's backward interpolation formula is given by

$$f(x) - \phi_n(x) = \frac{p(p+1)(p+2).......(p+n)}{(n+1)!} h^{n+1} y^{n+1} f(\varepsilon) \text{ Where } p = \frac{x - x_n}{h}$$

Examples:-

1. Find the melting point of the alloy containing 54% of lead, using appropriate interpolation formula

Percentage of lead(p)-X	50	60	70	80
Temperature $(Q°c)$-Y	205	225	248	274

Sol. The difference table is

X	Y	Δ	Δ^2	Δ^3
50=X₀	**205=Y₀**			
		20=$\Delta Y0$		
60	225		**3=$\Delta^2 Y_0$**	
		23		**0=$\Delta^3 Y_0$**
70	248		3	
		26		
80	274			

Let temperature = $f(x)$,X=54

$$x_0 + ph = 24, x_0 = 50, h = 10$$
$$50 + p(10) = 54\ (or)\ p = 0.4$$

By Newton's forward interpolation formula

$$f(x_0 + ph) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{n!}\Delta^3 y_0 + ---$$
$$f(54) = 205 + 0.4(20) + \frac{0.4(0.4-1)}{2!}(3) + \frac{(0.4)(0.4-1)(0.4-2)}{3!}(0)$$
$$= 205 + 8 - 0.36$$
$$= 212.64$$

Melting point = 212.64

2. Using Newton's forward interpolation formula, and the given table of values

X	1.1	1.3	1.5	1.7	1.9
$f(x)$	0.21	0.69	1.25	1.89	2.61

Obtain the value of $f(x)$ *when* $x = 1.4$

Sol.

X	$y = f(x)$	Δ	Δ^2	Δ^3	Δ^4
1.1	**0.21**				
		0.48			
1.3	0.69		**0.08**		
		0.56		0	
1.5	1.25		0.08		0
		0.64		0	
1.7	1.89		0.08		
		0.72			
1.9	2.61				

If we take $x_0 = 1.3 \ \ then \ \ y_0 = 0.69$,

$$\Delta y_0 = 0.56, \Delta^2 y_0 = 0.08, \Delta^3 y_0 = 0, L = 0.2, x = 1.3$$

$$x_0 + ph = 1.4(or)1.3 + p(0.2) = 1.4, p = \frac{1}{2}$$

Using Newton's interpolation formula

$$f(1.4) = 0.69 + \frac{1}{2} \times 0.56 + \frac{\frac{1}{2}\left(\frac{1}{2} - 1\right)}{2!} \times 0.08$$

$$= 0.69 + 0.28 - 0.01 = 0.96$$

3. The population of a town in the decimal census was given below. Estimate the population for the 1895

Year x	1891	1901	1911	1921	1931
Population of y	46	66	81	93	101

Sol. Putting $L=10, x_0=1891, x=1895$ in the formula $x=x_0+ph$ we obtain $p=2/5=0.4$

X	Y	Δ	Δ^2	Δ^3	Δ^4
1891=x0	46=y0				
		20$\Delta y0$			
1901	66		**-5**$\Delta^2 Y0$		
		15		**2**	
1911	81		-3		**-3**
		12		-1	
1921	93		-4		
		8			
1931	101				

$$y(1895)=46+(0.4)(20)+\frac{(0.4)(0.4-1)}{6}-(-5)$$
$$+\frac{(0.4-1)0.4(0.4-2)}{6}(2)$$
$$+\frac{(0.4)(0.4-1)(0.4-2)(0.4-3)}{24}$$
$$=54.45\ thousands$$

2.3. Gauss's Interpolation Formula:- We take x_0 as one of the specified of x that lies around the middle of the difference table and denote x_0-rh by $x-r$ and the corresponding value of y by $y-r$. Then the middle part of the forward difference table will appear as shown in the next page

X	Y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
x_{-4}	y_{-4}					
x_{-3}	y_{-3}	Δy_{-4}				
x_{-2}	y_{-2}	Δy_{-3}	$\Delta^2 y_{-4}$			
x_{-1}	y_{-1}	Δy_{-2}	$\Delta^2 y_{-3}$	$\Delta^3 y_{-4}$		
x_0	y_0	Δy_{-1}	$\Delta^2 y_{-2}$	$\Delta^3 y_{-3}$	$\Delta^4 y_{-4}$	
x_1	y_1	Δy_0	$\Delta^2 y_{-1}$	$\Delta^3 y_{-2}$	$\Delta^4 y_{-3}$	$\Delta^5 y_{-4}$
x_2	y_2	Δy_1	$\Delta^2 y_0$	$\Delta^3 y_{-1}$	$\Delta^4 y_{-2}$	$\Delta^5 y_{-3}$
x_3	y_3	Δy_2	$\Delta^2 y_1$	$\Delta^3 y_0$	$\Delta^4 y_{-1}$	$\Delta^5 y_{-2}$
x_4	y_4	Δy_3	$\Delta^2 y_2$	$\Delta^3 y_1$	$\Delta^4 y_0$	$\Delta^5 y_{-1}$

$$\Delta y_0 = \Delta y_{-1} + \Delta^2 y_{-1}$$
$$\Delta^2 y_0 = \Delta^2 y_{-1} + \Delta^3 y_{-1}$$
$$\Delta^3 y_0 = \Delta^3 y_{-1} + \Delta^4 y_{-1}$$
$$\Delta^4 y_0 = \Delta^4 y_{-1} + \Delta^5 y_{-1} ------(1) \ \ and$$
$$\Delta y_{-1} = \Delta y_{-2} + \Delta^2 y_{-2}$$
$$\Delta^2 y_{-1} = \Delta^2 y_{-2} + \Delta^3 y_{-2}$$
$$\Delta^3 y_{-1} = \Delta^3 y_{-2} + \Delta^4 y_{-2}$$
$$\Delta^4 y_{-1} = \Delta^4 y_{-2} + \Delta^5 y_{-2} ------(2)$$

By using the expressions (1) and (2), we now obtain two versions of the following Newton's forward interpolation formula

$$y_p = [y_0 + p(\Delta y_0) + \frac{p(p-1)}{2!}(\Delta^2 y_0) + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0$$
$$+ \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y_0 + ----] \quad \text{-----------------3}$$

Here y_p is the value of y at $x = x_p = x_0 + ph$,**P= (x-x₀)/h**

2.3.a. Gauss Forward Interpolation Formula:-

Substituting for $\Delta^2 y_0, \Delta^3 y_0, ...$ from (1)in the formula (3), we get

$$y_p = [y_0 + p(\Delta y_0) + \frac{p(p-1)}{2!}(\Delta^2 y_{-1} + \Delta^3 y_{-1}) + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_{-1}$$
$$+ \Delta^4 y_{-1} + \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y_{-1} + \Delta^5 y_{-1} + ----]$$

$$y_p = [y_0 + p(\Delta y_0) + \frac{p(p-1)}{2!}(\Delta^2 y_{-1}) + \frac{p(p+1)(p-1)}{3!}\Delta^3 y_{-1}$$
$$+ \frac{p(p+1)(p-1)(p-2)}{4!}(\Delta^4 y_{-1}) + ----]$$

Substituting $\Delta^4 y_{-1}$ *from* (2), this becomes

$$y_p = [y_0 + p(\Delta y_0) + \frac{p(p-1)}{2!}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}\Delta^3 y_{-1}$$
$$+ \frac{(p+1)(p-1)p(p-2)}{4!}(\Delta^4 y_{-2}) + ----] \quad \text{-----------------4}$$

Note:- we observe from the difference table that

$\Delta y_0 = \delta y_{1/2}, \Delta^2 y_{-1} = \delta^2 y_0, \Delta^3 y_{-1} = \delta^3 y_{1/2}, \Delta^4 y_{-2} = \delta^4 y_0$ and so on. Accordingly the formula

(4) can be written in the notation of central differences as given below

$$y_p = [y_0 + p\delta y_{1/2} + \frac{p(p-1)}{2!}\delta^2 y_0 + \frac{(p+1)p(p-1)}{3!}\delta^3 y_{1/2}$$
$$+ \frac{(p+1)(p-1)p(p-2)}{4!}\delta^4 y_0 + ----] \quad \text{-----------------5}$$

2.3.b. Gauss's Backward Interpolation formula:-

Let us substitute for $\Delta y_0, \Delta^2 y_0, \Delta^3 y_0$ ----- from (1) in the formula (3), thus we obtain

$$y_p = [y_0 + p(\Delta y_{-1} + \Delta^2 y_{-1}) + \frac{p(p-1)}{2!}(\Delta^2 y_{-1} + \Delta^3 y_{-1}) + \frac{(p-1)p(p-2)}{3!}(\Delta^3 y_{-1} + \Delta^4 y_{-1}) + \frac{(p-1)(p-2)p(p-3)}{4!}(\Delta^4 y_{-1} + \Delta^5 y_{-1}) + ----]$$

$$= [y_0 + p(\Delta y_{-1}) + \frac{(p+1)}{2!}p(\Delta^2 y_{-1}) + \frac{(p+1)p(p-1)}{3!}\Delta^3 y_{-1} + \frac{(p+1)p(p-1)(p-2)}{4!}(\Delta^4 y_{-1}) + ---]$$

Substituting for $\Delta^3 y_{-1}$ and $\Delta^4 y_{-1}$ from (2) this becomes

$$y_p = [y_0 + p(\Delta y_{-1}) + \frac{(p+1)p}{2!}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}(\Delta^3 y_{-1} + \Delta^4 y_{-2}) + \frac{(p+1)p(p-1)(p-2)}{4!}(\Delta^4 y_{-2} + \Delta^5 y_{-2}) + ----]$$

2.4. Lagrange's Interpolation Formula:-

Let $x_0, x_1, x_2, ... x_n$ be the $(n+1)$ values of x which are not necessarily equally spaced. Let $y_0, y_1, y_2 .... y_n$ be the corresponding values of $y = f(x)$ let the polynomial of degree n for the function $y = f(x)$ passing through the $(n+1)$ points $(x_0, f(x_0)), (x_1, f(x_1)) ---- (x_n, f(x_n))$ be in the following form

$$y = f(x) = a_0(x - x_1)(x - x_2)......(x - x_n) + a_1(x - x_0)(x - x_2).........(x - x_n) + a_2(x - x_0)(x - x_1).......(x - x_n) + ... + a_n(x - x_0)(x - x_1)......(x - x_{n-1}) \rightarrow (1)$$

Where $a_0, a_1, a_2 ....$ a^n are constants

Since the polynomial passes through $(x_0, f(x_0)), (x_1, f(x_1))......(x_n, f(x_n))$. The constants can be determined by substituting one of the values of $x_0, x_1, ...x_n$ *for* x in the above equation

Putting $x = x_0$ in (1) we get, $f(x_0) = a_0(x - x_1)(x_0 - x_2)(x_0 - x_n)$

$$\Rightarrow a_0 = \frac{f(x_0)}{(x - x_1)(x_0 - x_2)....(x_0 - x_n)}$$

Putting $x = x_1$ in (1) we get, $f(x_1) = a_1(x - x_0)(x_1 - x_2) - - - -(x_1 - x_n)$

$$\Rightarrow a_1 = \frac{f(x_1)}{(x_1 - x_0)(x_1 - x_2)....(x_1 - x_n)}$$

Similarly substituting $x = x_2$ in (1), we get

$$\Rightarrow a_2 = \frac{f(x_2)}{(x_2 - x_0)(x_2 - x_1)......(x_2 - x_n)}$$

Continuing in this manner and putting $x = x_n$ in (1) we

get $a_n = \dfrac{f(x_n)}{(x_n - x_0)(x_n - x_1) - - - -(x_n - x_{n-1})}$

Substituting the values of $a_0, a_1, a_2....a_n$, we get

$$f(x) = \frac{(x-x_1)(x-x_2).......(x-x_n)}{(x_0 - x_1)(x_0 - x_2).........(x_0 - x_n)} f(x_0) + \frac{(x-x_0)(x-x_2).....(x-x_n)}{(x_1 - x_0)(x_1 - x_2)....(x_1 - x_n)}$$

$$f(x_1) + \frac{(x-x_0)(x-x_1)(x-x_2).....(x-x_n)}{(x_2 - x_0)(x_2 - x_1)......(x_2 - x_n)} + ...f(x_2) + \frac{(x-x_0)(x-x_1).....(x-x_{n-1})}{(x_n - x_1)(x_n - x_2).....(x_n - x_{n-1})} f(x_n)$$

Examples:-

1. Using Lagrange's formula calculate $f(3)$ from the following table

x	0	1	2	4	5	6
$f(x)$	1	14	15	5	6	19

Sol. Given $x_0 = 0, x_1 = 1, x_2 = 2, x_3 = 4, x_5 = 6, x_4 = 5$

$$f(x_0) = 1, f(x_1) = 14, f(x_2) = 15, f(x_3) = 5, f(x_4) = 6, f(x_5) = 19$$

From langrage's interpolation formula

$$f(x)=\frac{(x-x_1)(x-x_2)(x-x_3)(x-x_4)(x-x_5)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)(x_0-x_4)(x_0-x_5)}f(x_0)$$
$$+\frac{(x-x_0)(x-x_2)(x-x_3)(x-x_4)(x-x_5)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)(x_1-x_4)(x_1-x_5)}f(x_1)$$
$$+\frac{(x-x_0)(x-x_1)(x-x_3)(x-x_4)(x-x_5)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)(x_2-x_4)(x_2-x_5)}f(x_2)$$

$$------------------$$

$$------------------$$

$$\frac{(x-x_0)(x-x_1)(x-x_2)(x-x_3)(x-x_4)}{(x_5-x_0)(x_5-x_1)(x_5-x_2)(x_5-x_3)(x_5-x_4)}f(x_5)$$

Here $x=3$ then

$$f(3)=\frac{(3-1)(3-2)(3-4)(3-5)(3-6)}{(0-1)(0-2)(0-4)(0-5)(0-6)}\times 1+$$
$$\frac{(3-0)(3-2)(3-4)(3-5)(3-6)}{(1-0)(1-2)(1-4)(1-5)(1-6)}\times 14+$$
$$\frac{(3-0)(3-1)(3-4)(3-5)(3-6)}{(2-0)(2-1)(2-4)(2-5)(2-6)}\times 15+$$

$$\frac{(3-0)(3-1)(3-2)(3-5)(3-6)}{(4-0)(4-1)(4-2)(4-5)(4-6)}\times 5+$$
$$\frac{(3-0)(3-1)(3-2)(3-4)(3-6)}{(5-0)(5-1)(5-2)(5-4)(5-6)}\times 6+$$

$$\frac{(3-0)(3-1)(3-2)(3-4)(3-5)}{(6-0)(6-1)(6-2)(6-4)(6-5)}\times 19$$
$$=\frac{12}{240}-\frac{18}{60}\times 14+\frac{36}{48}\times 15+\frac{36}{48}\times 5-\frac{18}{60}\times 6+\frac{12}{40}\times 19$$

$$=0.05-4.2+11.25+3.75-1.8+0.95$$

$$=10$$

f(3)=10

2. Find $f(3.5)$ using Lagrange method of 2^{nd} and 3^{rd} order degree polynomials.

x	1	2	3	4
$f(x)$	1	2	9	28

Sol: By lagrange's interpolation formula

$$f(x)=\sum_{k=0}^{n} f(x_k)\frac{(x-x_0).......(x-x_{k-1})(x-x_{k+1})(x-x_n)}{(x_k-x_0).......(x_k-x_{k-1}).......(x_k-x_n)}$$

$$f(x)=\frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}f(x_0)+$$

$$\frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}f(x_1)+$$
$$\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}f(x_2)+$$
$$\frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}f(x_3)+$$

$$\therefore f(3.5)=\frac{(3.5-2)(3.5-3)(3.5-4)}{(1-2)(1-3)(1-4)}(1)+\frac{(3.5-1)(3.5-3)(3.5-4)}{(2-1)(2-3)(2-4)}(2)+$$

$$\frac{(3.5-1)(3.5-2)(3.5-4)}{(3-1)(3-2)(3-4)}(9)+$$
$$\frac{(3.5-1)(3.5-2)(3.5-3)}{(4-1)(4-2)(4-3)}(28)+$$

- - - -

=0.0625+(-0.625)+8.4375+8.75

=16.625

$$f(x)=\frac{(x-2)(x-3)(x-4)}{-6}(1)+\frac{(x-1)(x-3)(x-4)}{2}(2)$$
$$+\frac{(x-1)(x-2)(x-4)}{(-2)}(9)+\frac{(x-1)(x-2)(x-3)}{6}(28)$$

$$=\frac{(x^2-5x+6)(x-4)}{-6}+(x^2-4x+3)(x-4)+\frac{(x^2-3x+2)}{-2}(x-4)(9)+\frac{(x^2-3x+2)}{6}(x-3)(28)$$

$$=\frac{x^3-9x^2+26x-24}{-6}+x^3-8x^2+9x-12+\frac{x^3-7x^2+14x-8}{-2}(9)+\frac{x^3-6x^2+11x-6}{6}(28)$$

$$=\frac{\left[-x^3+9x^2-26x+24+6x^3-48x^2+114x-72-27x^3+189x^2-378x+216+308x+28x^3-168x^2-168\right]}{6}$$

$$=\frac{6x^3-18x^2+18x}{6}\Rightarrow f(x)=x^3-3x^2+3x$$

$$\therefore f(3.5)=(3.5)^3-3(3.5)^2+3(3.5)=16.625$$

Gauss Formula Example:

1. Find y(25), given that $y_{20} = 24, y_{24} = 32$, $y_{28} = 35$, $y_{32} = 40$ using Gauss forward difference formula :

Solution: Given

X	20	24	28	32
Y	24	32	35	40

By Gauss Forward difference formula

$$y_p=[y_0+p(\Delta y_0)+\frac{p(p-1)}{2!}\Delta^2 y_{-1}+\frac{(p+1)p(p-1)}{3!}\Delta^3 y_{-1}$$
$$+\frac{(p+1)(p-1)p(p-2)}{4!}(\Delta^4 y_{-2})+----]\rightarrow(4)$$

We take x_0= 24 as origin.

X_0 = 24, h = 4, x = 25, p = $(x-x_0)$/ h, p = (25-24)/4 = 0.25

Gauss Forward difference table is

X	Y	Δy	$\Delta^2 y$	$\Delta^3 y$
20=x-1	24=y-1			
24=x0	**32=y0**	Δy_{-1}= 8		
28=x1	35=y1	Δy_0 = 3	$\Delta^2 y_{-1}$= -5	
32=x2	40=y2	Δy_1 = 5	$\Delta^2 y_0$ = 2	$\Delta^3 y_{-1}$ = 7

By gauss forward interpolation Formula

We get $y(25) = 32 + 0.25(3) + (\frac{0.25)(0.25-1)}{2})(-5) + \frac{(0.25+1)(0.25)(0.25-1)}{6}(7) = 32 + 0.75$

$+0.46875 - 0.2734 = 32.945$

$Y(25) = 32.945$.

2. Example:

Use Gauss Backward interpolation formula to find f(32) given that f(25) = 0.2707, f(30) = 0.3027, f(35) =0 .3386 , f(40) =0 .3794.

Solution: let x_0 = 35 and difference table is

X	Y	Δy	$\Delta^2 y$	$\Delta^3 y$
25=x-2	0.2707=y-2			
30=x-1	0.3027=y-1	0.032		
35=x0	**0.3386=y0**	**0.0359**	**0.0039**	
40=x1	0.3794=y1	0.0408	0.0049	0.0010

From the table y_0 = 0.3386 ,h=5,P=(x-x0)/h=(32-35)/5=-0.6

Δy_{-1} = 0.0359, $\Delta^2 y_{-1}$ = 0.0049, $\Delta^3 y_{-2}$ = 0.0010, x_p = 32 p = (x_p- x_0)/h = (32-35)/5 = -0.6

By Gauss Backward difference formula

$$y_p = [y_0 + p(\Delta y_{-1}) + \frac{(p+1)p}{2!}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}(\Delta^3 y_{-1} + \Delta^4 y_{-2})$$
$$+ \frac{(p+1)p(p-1)(p-2)}{4!}(\Delta^4 y_{-2} + \Delta^5 y_{-2}) + ----]$$

f(32) = 0.3386 + (-0.6)(0.0359) + (-0.6)(-0.6+1)(0.0049)/2 + (-0.6)(0.36-1)(0.00010)/6 =0 .3165

2.5 Stirling's Formula:

Stirling's formula is arithmetic mean of Gauss forward interpolation and Gauss Backward Interpolation formulae

We know that from Gauss forward formula

$$y_p = [y_0 + p(\Delta y_0) + \frac{p(p-1)}{2!}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}\Delta^3 y_{-1} + \frac{(p+1)(p-1)p(p-2)}{4!}(\Delta^4 y_{-2}) + ----] \rightarrow (4)$$

And from Gauss backward formula

$$y_p = [y_0 + p(\Delta y_{-1}) + \frac{(p+1)p}{2!}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}(\Delta^3 y_{-1} + \Delta^4 y_{-2}) + \frac{(p+1)p(p-1)(p-2)}{4!}(\Delta^4 y_{-2} + \Delta^5 y_{-2}) + ----] \quad ----(5)$$

From 4,5, we found arithmetic mean the Stirling's formula is defined as

$P=\frac{X-X0}{h}$, where $h=x_1-x_0$

$Y(x) = y_0 + P\left[\frac{\Delta Y-1+\Delta Y0}{2}\right] + \frac{P^2}{2}\Delta^2 Y_{-1} + \frac{[P(P^2-1)]}{3!}\left[\frac{\Delta^3 Y-2+\Delta^3 Y-1}{2}\right] + \frac{P^2(P^2-1)}{4!}\Delta^4 Y_{-2}$ +----

2.5 Bessel's Formula:

Y=f(x) is a function with data (x_i, y_i) with $P=\frac{X-X0}{h}$, where $h=x_1-x_0$ then Bessel's formula is defined as follows

$Y(X)= Y_0+P\Delta Y0 + \frac{P(P-1)}{2!}\left[\frac{\Delta^2 Y0+\Delta^2 Y-1}{2}\right] + \frac{(P-\frac{1}{2})P(P-1)}{3!}\Delta^3 Y_{-1} + \frac{P(P-1)(P+1)(P-2)}{4!}\left[\frac{\Delta^4 Y-2+\Delta^4 Y-1}{2}\right] + ---$

Examples:

1. Using Striling's formula, compute f(1.22) from the following data

X	1.0	1.1	**1.2**	1.3	1.4
F(x)	0.841	0.891	0.932	0.963	0.985

Sol. Chose X0=1.2 is origin and length h=0.1 and P=$\frac{X-X0}{h}=\frac{1.22-1.2}{0.1}$ =0.2

Next we construct central difference table by using above data and evaluate required value by Stirling's formula

X	Y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
x_{-2} =1.0	y_{-2} =0.841				
x_{-1} =1.1	y_{-1} =0.891	Δy_{-2} =0.05			
x_0 =1.2	y_0 =0.932	Δy_{-1} =0.041	$\Delta^2 y_{-2}$ =-0.009		
x_1 =1.3	y_1 =0.963	Δy_0 =0.031	$\Delta^2 y_{-1}$ =-0.01	$\Delta^3 y_{-2}$ =-0.001	
x_2 =1.4	y_2 =0.985	Δy_1 =0.022	$\Delta^2 y_0$ =-0.009	$\Delta^3 y_{-1}$ =0.001	$\Delta^4 y_{-2}$ =0.02

Use Stirling's formula

$$Y(x) = y_0 + P\left[\frac{\Delta Y-1+\Delta Y0}{2}\right] + \frac{P^2}{2}\Delta^2 Y_{-1} + [P(P^2-1)]/3!\left[\frac{\Delta^3 Y-2+\Delta^3 Y-1}{2}\right] + \frac{P^2(P^2-1)}{4!}\Delta^4 Y_{-2} + \text{----}$$

Then $Y(1.22) = 0.932+0.2\frac{(0.041+0.031)}{2} + \frac{0.04}{2!}(-0.01) + \frac{0.2(0.04-1)}{4!}(0.002) = 0.93899$

2. Find f(16) by Stirling's formula from the following table

x	0	5	10	15	20	25	30
F(x)	0	0.0875	0.1763	0.2679	0.364	0.4663	0.5774

Examples:

1. Use Bessel's formula to compute f(1.95) from the following data

X	1.7	1.8	1.9	2.0	2.1	2.2	2.3
F(X)	2.979	3.144	3.283	3.391	3.463	3.997	4.491

Sol. Choose the origin at X_0 =2.0, given h=0.1 and P= $\frac{x-x0}{h}=\frac{1.95-2.0}{0.1}$ =-0.5

Next by using Bessel's formula and central difference table we can evaluate the required solution

X	Y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
x_{-3} =1.7	y_{-3} =2.979					
x_{-2} =1.8	y_{-2} =3.144	Δy_{-3} =0.165				
x_{-1} =1.9	y_{-1} =3.283	Δy_{-2} =0.139	$\Delta^2 y_{-3}$ =-0.026			
x_0 **=2.0**	y_0 **=3.391**	Δy_{-1} =0.108	$\Delta^2 y_{-2}$ =-0.031	$\Delta^3 y_{-3}$ =-0.005		
x_1 =2.1	y_1 =3.463	Δy_0 =0.072	$\Delta^2 y_{-1}$ =-0.036	$\Delta^3 y_{-2}$ =-0.005	$\Delta^4 y_{-3}$ =0	
x_2 =2.2	y_2 =3.997	Δy_1 =0.53	$\Delta^2 y_0$ =0.462	$\Delta^3 y_{-1}$ =0.498	$\Delta^4 y_{-2}$ =0.503	$\Delta^5 y_{-3}$ =0.503
x_3 =2.3	y_3 =4.491	Δy_2 =0.494	$\Delta^2 y_1$ =-0.04	$\Delta^3 y_0$ =-0.502	$\Delta^4 y_{-1}$ =-1	$\Delta^5 y_{-2}$ =-1.503

Bessel's formula $\Delta^6 y_{-3} = -2.006$

Y(X)= $Y_0 + P\Delta Y0 + \frac{P(P-1)}{2!}\left[\frac{\Delta^2 Y0+\Delta^2 Y-1}{2}\right] + \frac{(P-\frac{1}{2})P(P-1)}{3!}\Delta^3 Y_{-1} + \frac{P(P-1)(P+1)(P-2)}{4!}\left[\frac{\Delta^4 Y-2+\Delta^4 Y-1}{2}\right] + - - -$

Y(1.95) = 3.391+(-0.5)(0.072) + $\frac{(-0.5)(-0.5-1)(-0.036+0.462)}{2.2} + \frac{(-0.5-0.5)(-0.5)(-0.5-1)(-0.5-2)(0.503-1)}{24.2}$

Y(1.95) = 3.3629

2. Compute Y(25) by using Bessel's formula to the following table

X	20	24	28	32
Y	2854	3162	3544	3992

9. Practice Quiz

1. Newton's backward Interpolation formula is .... [a]

a.

$$y_p = [y_0 + p(\Delta y_{-1}) + \frac{(p+1)p}{2!}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}(\Delta^3 y_{-1} + \Delta^4 y_{-2}) + \frac{(p+1)p(p-1)(p-2)}{4!}(\Delta^4 y_{-2} + \Delta^5 y_{-2}) + ----]$$

b. $y+y_0$

c. y_0

d. None

2. The ....Interpolation formula is used to estimate y, if the x-values are unequally spaced. [c]

a. Newton formula

b. Gauss formula

c. Lagrange's formula

d. Bessel's formula

3. Averaging Operator formula [d]

a. Δ

b. ∇

c. U

d. $\mu y_r = \frac{1}{2}[y_{r+1/2} + y_{r-1/2}]$

4. The relation between Δ *and* E [c]

a. $\Delta = E$

b. $\nabla = E$

c. $\Delta = E - 1$

d. $\Delta = E + 1$

5. Find the missing term in the following data [c]

x	0	1	2	3	4
Y	1	3	9	-	81

a. 10

b. 19

c. 27

d. 0

6. The relation between ∇ and E^{-1} [a]

a. $\nabla \equiv 1 - E^{-1}$

b. $\delta \equiv E^{1/2} - E^{-1/2}$

c. $\mu = \frac{1}{2}\left(E^{1/2} + E^{-1/2}\right)$

d. $\mu^2 \equiv 1 + \frac{1}{4}\delta^2$

7. The relation between δ *and* E [b]

a. $\nabla \equiv 1 - E^{-1}$

b. $\delta \equiv E^{1/2} - E^{-1/2}$

c. $\mu = \frac{1}{2}\left(E^{1/2} + E^{-1/2}\right)$

d. $\mu^2 \equiv 1 + \frac{1}{4}\delta^2$

8. The relation between μ *and* E [c]

a. $\nabla \equiv 1 - E^{-1}$

b. $\delta \equiv E^{1/2} - E^{-1/2}$

c. $\mu = \frac{1}{2}\left(E^{1/2} + E^{-1/2}\right)$

d. $\mu^2 \equiv 1+\frac{1}{4}\delta^2$

9. The relation between $\mu \; and \; \delta$ [d]

a. $\nabla \equiv 1-E^{-1}$

b. $\delta \equiv E^{1/2}-E^{-1/2}$

c. $\mu = \frac{1}{2}\left(E^{1/2}+E^{-1/2}\right)$

d. $\mu^2 \equiv 1+\frac{1}{4}\delta^2$

10. The relation between E and D [d]

a. $\nabla \equiv 1-E^{-1}$

b. $\delta \equiv E^{1/2}-E^{-1/2}$

c. $\mu = \frac{1}{2}\left(E^{1/2}+E^{-1/2}\right)$

d. $E = e^{hD}$

10. Assignments

S.No	Question	BL	CO
1	Using Newton's forward interpolation formula, and the given table of values. X: 1.1, 1.3, 1.5, 1.7, 1.9 F(X): 0.21, 0.69, 1.25, 1.89, 2.61 Obtain the value of $f(x)$ $when$ $x = 1.4$	1	2
2	Use Gauss back ward interpolation formula to find $f(32)$ given that $f(25) = 0.2707, f(30) = 0.3027, f(35) = 0.3386, f(40) = 0.3794.$	1	2
3	Evaluate $f(10)$ given $f(x) = 168,192,336$ at $x = 1,7,15$ respectively. Use Lagrange interpolation.	5	2
4	Apply Bessel's formula to obtain $f(25)$ given $f(20)=2854,\ f(24)=3162,\ f(12)=3544,\ f(12)=3992$.	1	2
5	Apply Stirling's formula to obtain $f(35)$ given $f(20)=512,\ f(30)=439,\ f(40)=346,\ f(50)=243$.	1	2

11. Part A- Question & Answers

S.No	Question& Answers	BL	CO
1	**Distinguish between interpolation and extrapolation** Sol. Interpolation: It is the estimation for some such values which lie inside the given Values. Extrapolation: It is the estimation for some such values which lie outside the given Values.	4	2
2	**Write relation between E and Δ.** Sol. $\Delta f(x) = f(x+h) - f(x)$ Forward Definition $= Ef(x) - f(x)$ Shift Definition $= (E-1) f(x)$	1	2
3	**Prove that $(1+\Delta)(1-\nabla) = 1$** Sol. We know that $(1+\Delta) = E, (1-\nabla) = E^{-1}$ $(1+\Delta)(1-\nabla) = EE^{-1} = 1$	5	2
4	**Evaluate $\Delta \tan^{-1} x$** Sol. $\Delta \tan^{-1} x = \tan^{-1}(x+h) - \tan^{-1} x$ $= \tan^{-1}\left(\frac{x+h-x}{1+(x+h)x}\right) = \tan^{-1}\left(\frac{h}{1+x^2+hx}\right)$	5	2
5	**Evaluate Δe^x if h=1.** Sol. We know that $\Delta e^x = e^{x+h} - e^x$ $= e^{x+1} - e^x$ $= (e-1)e^x$	5	2
6	**Evaluate $\Delta^{10}\left[(1-ax)(1-bx^2)(1-cx^3)(1-dx^4)\right]$.** Sol. Let $f(x) = \Delta^{10}\left[(1-ax)(1-bx^2)(1-cx^3)(1-dx^4)\right]$ f(x) is a polynomial of degree 10 and the coefficient of x^{10} is abcd. $\Delta^{10}\left[(1-ax)(1-bx^2)(1-cx^3)(1-dx^4)\right] = abcd\Delta^{10}x^{10} = abcd10!$.	5	2
7	**Write Newton-Gregory forward interpolation formula.** Sol. $y = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + ---------$	1	2
8	**Write Newton-Gregory backward interpolation formula.** Sol. $y = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + ----------$	1	2
9	**State Langrage's interpolation formula.** Sol. Let x_1, x_2, x_3-------------x_n be the values of x which are not equally spaced and y_1, y_2, y_3------y_n be the Corresponding values of y. Thus Lagrange's interpolation formula is	1	2

$$y = \frac{(x-x_2)(x-x_3)----(x-x_n)}{(x_1-x_2)(x_1-x_3)----(x_1-x_n)} y_1 + \frac{(x-x_1)(x-x_3)----(x-x_n)}{(x_2-x_1)(x_2-x_3)----(x_2-x_n)} y_2 + \text{--------} + \frac{(x-x_1)(x-x_2)----(x-x_{n-1})}{(x_n-x_1)(x_n-x_2)----(x_n-x_{n-1})} y_n .$$

10	**Write Stirling's Formula** Sol. $Y(x) = y_0 + P\left[\frac{\Delta Y-1+\Delta Y0}{2}\right] + \frac{P^2}{2}\Delta^2 Y_{-1} + \frac{[P(P^2-1)]}{3!}\left[\frac{\Delta^3 Y-2+\Delta^3 Y-1}{2}\right] + \frac{P^2(P^2-1)}{4!}\Delta^4 Y_{-2} + \text{---}\infty$	1	2

12. Part B- Questions

S.No	Question	BL	CO
1	For $x = 0,1,2,4,5; f(x) = 1,14,15,5,6. find\ f(3)$ using forward difference table.	1	2
2	Find the values of $cos1.747$ using the values given in the table below X: 1.70, 1.74, 1.78, 1.82, 1.86 F(X): 0.9916, 0.9857, 0.9781, 0.9691, 0.9584	1	2
3	Using Newton's forward interpolation formula, and the given table of values. X: 1.1, 1.3, 1.5, 1.7, 1.9 F(X): 0.21, 0.69, 1.25, 1.89, 2.61 Obtain the value of $f(x)\ when\ x = 1.4$	1	2
4	Use Gauss back ward interpolation formula to find $f(32)$ given that $f(25) = 0.2707, f(30) = 0.3027, f(35) = 0.3386, f(40) = 0.3794.$	1	2
5	Use Lagrange interpolation. Evaluate $f(10)$ given $f(x) = 168,192,336\ at\ x = 1,7,15$ respectively.	5	2
6	Find the unique polynomial $p(x)$ of degree 2 or less such that $p(x) = 1, p(3) = 27, p(4) = 64$ using Lagrange's interpolation formula.	1	2
7	Given $f(2) = 10, f(1) = 8, f(0) = 5, f(-1) = 10\ estimate\ f(\frac{1}{2})$ by using Gauss forward formula	6	2
8	Using Lagrange's Interpolation formula, find $y(10)$from the following table X: 5, 6, 9, 11 Y: 12, 13, 14, 16 Fit the second difference of the polynomial $x^4 - 12x^3 + 42x^2 - 30x + 9$ with interval of differencing $h = 2$	1	2
9	Apply Bessel's formula to obtain $f(25)$ given $f(20) = 2854, f(24) = 3162, f(12) = 3544, f(12) = 3992.$	1	2

$$\int_1^2 \frac{1}{x}\,dx = \frac{1}{3}[(1+1.4142)+3(1.0004+1.0062+1.0943+1.2175)$$
$$+2(1.0301)]$$

=1.0894

Numerical solutions of ordinary differential equations

1. The important methods of solving ordinary differential equations of first order numerically are as follows
    1) Taylor's series method
    2) Picard's method
    3) Euler's method
    4) Modified Euler's method of successive approximations
    5) Runge- kutta method

To describe various numerical methods for the solution of ordinary differential eqn's, we consider the general 1st order differential eqn

Given O.D.Eqn. dy/dx=f(x,y)-------(1)

with the initial condition $y(x_0)=y_0$, $X_1=X_0+h$, $X_2=X_1+h$,we have to evaluate Y_1,Y_2---- etc

The methods will yield the solution in one of the two forms:

i) A series for y in terms of powers of x, from which the value of y can be obtained by direct substitution.

ii) A set of tabulated values of y corresponding to different values of x

The methods of Taylor and Picard belong to class(i)

The methods of Euler, Runge - kutta method, Adams, Milne etc, belong to class (ii)

3.4 TAYLOR'S SERIES METHOD

To find the numerical solution of the differential equation

$\frac{dy}{dx} = f(x,y) \rightarrow (1)$

With the initial condition $y(x_0) = y_0 \rightarrow (2)$

$y(x)$ can be expanded about the point x_0 in a Taylor's series in powers of $(x-x_0)$ as

$$y(x) = y(x_0) + \frac{(x-x_0)}{1} y'(x_0) + \frac{(x-x_0)^2}{2!} y''(x_0) + \ldots\ldots\ldots\ldots + \frac{(x-x_0)^n}{n!} y^n(x_0) \rightarrow (3)$$

In equ3, $y(x_0)$ is known from I.C equ2. The remaining coefficients $y'(x_0), y''(x_0), \ldots\ldots\ldots y^n(x_0)$ etc are obtained by successively differentiating equ1 and evaluating at x_0. Substituting these values in equ3, $y(x)$ at any point can be calculated from equ3. Provided $h = x - x_0$ is small.

When $x_0 = 0$, then Taylor's series equ3 can be written as

$$y(x) = y(0) + x.y'(0) + \frac{x^2}{2!} y''(0) + \ldots\ldots + \frac{x^n}{n!} y^n(0) + \ldots\ldots\ldots \rightarrow (4)$$

1. Using Taylor's expansion evaluate the value of $y' - 2y = 3e^x, y(0) = 0$**, at a)** $x = 0.2$

b) Compare the numerical solution obtained with exact solution .

Sol: Given equation can be written as $2y + 3e^x = y', y(0) = 0$ $x_0=0$, $y_0=0$

Differentiating repeatedly w.r.t to 'x' and evaluating at $x = 0$

$$y'(x) = 2y + 3e^x, y'(0) = 2y(0) + 3e^0 = 2(0) + 3(1) = 3$$
$$y''(x) = 2y' + 3e^x, y''(0) = 2y'(0) + 3e^0 = 2(3) + 3 = 9$$
$$y'''(x) = 2.y''(x) + 3e^x, y'''(0) = 2y''(0) + 3e^0 = 2(9) + 3 = 21$$
$$y^{iv}(x) = 2.y'''(x) + 3e^x, y^{iv}(0) = 2(21) + 3e^0 = 45$$
$$y^{v}(x) = 2.y^{iv} + 3e^x, y^{v}(0) = 2(45) + 3e^0 = 90 + 3 = 93$$

In general, $y^{(n+1)}(x) = 2.y^{(n)}(x) + 3e^x$ or $y^{(n+1)}(0) = 2.y^{(n)}(0) + 3e^0$

The Taylor's series expansion of $y(x)$ about $x_0 = 0$ is

$$y(x) = y(0) + xy'(0) + \frac{x^2}{2!}y''(0) + \frac{x^3}{3!}y'''(0) + \frac{x^4}{4!}y''''(0) + \frac{x^5}{5!}y'''''(0) + ...$$

Substituting the values of $y(0), y'(0), y''(0), y'''(0), ...$

$$y(x) = 0 + 3x + \frac{9}{2}x^2 + \frac{21}{6}x^3 + \frac{45}{24}x^4 + \frac{93}{120}x^5 + ...$$

$$y(x) = 3x + \frac{9}{2}x^2 + \frac{7}{2}x^3 + \frac{15}{8}x^4 + \frac{31}{40}x^5 + ... \rightarrow \text{equ1}$$

Now put $x = 0.1$ in equ1

$$y(0.1) = 3(0.1) + \frac{9}{2}(0.1)^2 + \frac{7}{2}(0.1)^3 + \frac{15}{8}(0.1)^4 + \frac{31}{40}(0.1)^5 = 0.34869$$

Now put $x = 0.2$ in equ1

$$y(0.2) = 3(0.2) + \frac{9}{2}(0.2)^2 + \frac{7}{2}(0.2)^3 + \frac{15}{8}(0.2)^4 + \frac{31}{40}(0.2)^5 = 0.811244$$

$$y(0.3) = 3(0.3) + \frac{9}{2}(0.3)^2 + \frac{7}{2}(0.3)^3 + \frac{15}{8}(0.3)^4 + \frac{31}{40}(0.3)^5 = 1.41657075$$

Analytical Solution:

The exact solution of the equation $\frac{dy}{dx} = 2y + 3e^x$ with $y(0) = 0$ can be found as follows

$\frac{dy}{dx} - 2y = 3e^x$ Which is a linear in y.

Here $P = -2, Q = 3e^x$

I.F = $\int_e^{pdx} = \int_e^{-2dx} = e^{-24}$

General solution is $y.e^{-2x} = \int 3e^x.e^{-2x}dx + c = -3e^{-x} + c$,dividing by e^{-2x} on b.s.

$\therefore y = -3e^x + ce^{2x}$ where $x = 0, y = 0$, $0 = -3 + c \Rightarrow c = 3$

The particular solution is $y = 3e^{2x} - 3e^x$ or $y(x) = 3e^{2x} - 3e^x$

Put $x = 0.1$ in the above particular solution,

$$y = 3.e^{0.2} - 3e^{0.1} = 0.34869$$

Similarly put $x = 0.2$

$y = 3e^{0.4} - 3e^{0.2} = 0.811265$

put $x = 0.3$

$y = 3e^{0.6} - 3e^{0.3} = 1.416577$

There is negligible error between numerical solution and analytical solution.

2. Using Taylor's series method, solve the equation $\frac{dy}{dx} = x^2 + y^2$ for $x = 0.4$ given that $y = 0$ when $x = 0$

Sol: Given that $\frac{dy}{dx} = x^2 + y^2$ and $y = 0$ when $x = 0$ i.e. $y(0) = 0$

Here $y_0 = 0$, $x_0 = 0$

Differentiating repeatedly w.r.t 'x' and evaluating at $x = 0$

$$y'(x) = x^2 + y^2, y'(0) = 0 + y^2(0) = 0 + 0 = 0$$
$$y''(x) = 2x + y'.2y, y''(0) = 2(0) + y'(0)2.y = 0$$
$$y'''(x) = 2 + 2yy'' + 2y'.y', y'''(0) = 2 + 2.y(0).y''(0) + 2.y'(0)^2 = 2$$
$$y''''(x) = 2.y.y''' + 2.y''.y' + 4.y''.y', y''''(0) = 0$$

The Taylor's series for f(x) about $x_0 = 0$ is

$$y(x) = y(0) + xy'(0) + \frac{x^2}{2!}y''(0) + \frac{x^3}{3!}y'''(0) + \frac{x^4}{4!}y''''(0) + \ldots$$

Substituting the values of $y(0), y'(0), y''(0), \ldots..$

$y(x) = 0 + x(0) + 0 + \frac{2x^3}{3!} + 0 + \ldots\ldots.. = \frac{x^3}{3} +$ (Higher order terms are neglected)

$$\therefore y(0.4) = \frac{(0.4)^3}{3} = \frac{0.064}{3} = 0.02133$$

3. Solve $y' = x - y^2, y(0) = 1$ using Taylor's series method and compute y(0.1),y(0.2)

Sol: Given that $y' = x - y^2, y(0) = 1$

Here $y_0 = 1$, $x_0 = 0$

Differentiating repeatedly w.r.t 'x' and evaluating at x=0

$$y'(x) = x - y^2, y'(0) = 0 - y(0)^2 = 0 - 1 = -1$$
$$y''(x) = 1 - 2y.y', y''(0) = 1 - 2.y(0)y'(0) = 1 - 2(-1) = 3$$
$$y'''(x) = 1 - 2yy' - 2(y')^2, y'''(0) = -2.y(0).y''(0) - 2.(y'(0))^2 = -6 - 2 = -8$$
$$y''''(x) = -2.y.y''' - 2.y''.y' - 4.y''.y', y''''(0) = -2.y(0).y'''(0) - 6.y''(0).y'(0) = 16 + 18 = 34$$

The Taylor's series for f(x) about $x_0 = 0$ is

$$y(x) = y(0) + \frac{x}{1!}y^1(0) + \frac{x^2}{2!}y^{11}(0) + \frac{x^3}{3!}y^{111}(0) + ...$$

Substituting the value of $y(0)$, $y^1(0)$, $y^{11}(0)$,.....

$$y(x) = 1 - x + \frac{3}{2}x^2 - \frac{8}{6}x^3 + \frac{34}{24}x^4 + ...$$

$$y(x) = 1 - x + \frac{3}{2}x^2 - \frac{4}{3}x^3 + \frac{17}{12}x^4 + ... \quad \rightarrow (1)$$

now put x = 0.1 in (1)

$$y(0.1) = 1 - 0.1 + \frac{3}{2}(0.1)^2 + \frac{4}{3}(0.1)^3 + \frac{17}{12}(0.1)^4 + ...$$

= 0.91380333 $\simeq$ 0.91381

Similarly put x = 0.2 in (1)

$$y(0.2) = 1 - 0.2 + \frac{3}{2}(0.2)^2 - \frac{4}{3}(0.2)^3 + \frac{17}{12}(0.2)^4 + ...$$

= 0.8516.

4. Solve $y^1 = x^2 - y$, $y(0) = 1$, using Taylor's series method and compute y(0.1), y(0.2), y(0.3) and y(0.4) (correct to 4 decimal places).

Sol. Given that $y^I = x^2 - y$ and $y(0) = 1$

Here $x_0 = 0$, $y_0 = 1$ or y =1 when x=0

Differentiating repeatedly w.r.t 'x' and evaluating at x = 0.

$Y^I(x) = x^2 - y$, $y^I(0) = 0 - 1 = -1$

$y^{II}(x) = 2x - y^I$, $y^{II}(0) = 2(0) - y^I(0) = 0 - (-1) = 1$

$y^{III}(x) = 2 - y^{II}$, $y^{III}(0) = 2 - y^{II}(0) = 2 - 1 = 1$,

$y^{IV}(x) = - y^{III}$, $y^{IV}(0) = -y^{III}(0) = -1$.

The Taylor's servies for f(x) about $x_0 = 0$ is

$$y(x) = y(0) + \frac{x}{1!}y^I(0) + \frac{x^2}{2!}y^{II}(0) + \frac{x^3}{3!}y^{III}(0) + \frac{x^4}{4!}y^{IV}(0) + \ldots\ldots$$

substituting the values of $y(0)$, $y^1(0)$, $y^{11}(0)$, $y^{111}(0)$,…….

$$y(x) = 1 + x(-1) + \frac{x^2}{2}(1) + \frac{x^3}{6}(1) + \frac{x^4}{24}(-1) + \ldots\ldots$$

$$y(x) = 1 - x + \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^4}{24} + \ldots\ldots \quad \rightarrow(1)$$

Now put x = 0.1 in (1),

$$y(0.1) = 1 - 0.1 + \frac{(0.1)^2}{2} + \frac{(0.1)^3}{6} - \frac{(0.1)^4}{24} + \ldots.$$

= 1 – 0.1 + 0.005 + 0.01666 – 0.0000416 -0.905125 ~ 0.9051

(4 decimal places)

Now put x = 0.2 in eq (1),

$$y(0.2) = 1 - 0.2 + \frac{(0.2)^2}{2} + \frac{(0.2)^3}{6} - \frac{(0.2)^4}{64}$$

= 1 – 0.2 + 0.02 + 0.001333 – 0.000025

= 1.021333 – 0.200025

= 0.821308 ~ 0.8213 (4 decimals)

Similarly y(0.3) = 0.7492 and y (0.4) = 0.6897 (4 decimal places).

5. Solve $\frac{dy}{dx}$ -1 = xy and y(0) = 1 using Taylor's series method and compute y(0.1).

Sol. Given that $\frac{dy}{dx} - 1 = xy$ and $y(0) = 1$

Here $\frac{dy}{dx} = 1 + xy$ and $y_0 = 1$, $x_0 = 0$.

Differentiating repeatedly w.r.t 'x' and evaluating at $x_0 = 0$

$y^I(x) = 1 + xy$, $\quad y^I(0) = 1+0(1) = 1$.

$y^{II}(x) = x.y' + y$, $\quad y^{II}(0) = 0+1=1$

$y^{III}(x) = x.y'' + y^I + y^I$, $\quad y^{III}(0) = 0.(1) + 2\,(1) = 2$

$y^{IV}(x) = xy^{III} + y^{II} + 2y^{II}$, $\quad y^{IV}(0) = 0+3(1) = 3$.

$y^V(x) = xy^{IV} + y^{III} + 2y^{III}$, $\quad y^V(0) = 0 + 2 + 2(3) = 8$

The Taylor series for f(x) about $x_0 = 0$ is

$$y(x) = y(0) + x.y^{I}(0) + \frac{x^2}{2!} y^{II}(0) + \frac{x^3}{3!} y^{III}(0) + \frac{x^4}{4!} y^{IV}(0) + \frac{x^5}{5!} y^{V}(0) + \ldots..$$

Substituting the values of $y(0)$, $y^{I}(0)$, $y^{II}(0)$, ….

$$y(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6}(2) + \frac{x^4}{24}(3) + \frac{x^5}{120}(8) + \ldots.$$

$$y(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{8} + \frac{x^5}{15} + \ldots. \quad \rightarrow (1)$$

Now put x = 0.1 in equ (1),

$$y(0.1) = 1 + 0.1 + \frac{(0.1)^2}{2} + \frac{(0.1)^3}{3} + \frac{(0.1)^4}{8} + \frac{(0.1)^5}{15} + \ldots..$$

= 1 + 0.1 +0.005 + 0.000333 + 0.0000125 + 0.0000006

= 1.1053461

H.W

6. Given the differential equ $y^1 = x^2 + y^2$, $y(0) = 1$.Obtain y(0.25), and y(0.5) by Taylor's Series method.

Ans: 1.3333, 1.81667

7. Solve $y^1 = xy^2 + y$, $y(0) =1$ using Taylor's series method and compute y(0.1) and y(0.2).

Ans: 1.111, 1.248.

Note: We know that the Taylor's expansion of y(x) about the point x_0 in a power of $(x - x_0)$is.

$$y(x) = y(x_0) + \frac{(x-x_0)}{1!} y^{I}(x_0) + \frac{(x-x_0)^2}{2!} y^{II}(x_0) + \frac{(x-x_0)^3}{3!} y^{III}(x_0) + \ldots \rightarrow (1)$$

Or

$$y(x) = y_0 + \frac{(x-x_0)}{1!} y_0^{I} + \frac{(x-x_0)^2}{2!} y_0^{II} + \frac{(x-x_0)^3}{3!} y_0^{III} + \ldots..$$

If we let $x - x_0 = h$. (i.e. $x = x_0 + h = x_1$) we can write the Taylor's series as

$$y(x) = y(x_1) = y_0 + \frac{h}{1!} y_0^I + \frac{h^2}{2!} y_0^{II} + \frac{h^3}{3!} y_0^{III} + \frac{h^4}{4!} y_0^{IV} + \ldots.$$

$$\text{i.e. } y_1 = y_0 + \frac{h}{1!} y_0^I + \frac{h^2}{2!} y_0^{II} + \frac{h^3}{3!} y_0^{III} + \frac{h^{IV}}{4!} y_0^{IV} + \ldots.. \quad \rightarrow (2)$$

Similarly expanding y(x) in a Taylor's series about $x = x_1$. We will get.

$$y_2 = y1 + \frac{h}{1!} y_1^I + \frac{h^2}{2!} y_1^{II} + \frac{h^3}{3!} y_1^{III} + \frac{h^4}{4!} y_1^{IV} + \ldots\ldots. \quad \rightarrow (3)$$

Similarly expanding y(x) in a Taylor's series about $x = x_2$ We will get.

$$y_3 = y_2 + \frac{h}{1!} y_2^I + \frac{h^2}{2!} y_2^{II} + \frac{h^3}{3!} y_2^{III} + \frac{h^4}{4!} y_2^{IV} + \ldots\ldots \quad \rightarrow (4)$$

In general, Taylor's expansion of y(x) at a point $x = x_n$ is

$$y_{n+1} = y_n + \frac{h}{1!} y_n^I + \frac{h^2}{2!} y_n^{II} + \frac{h^3}{3!} y_n^{III} + \frac{h^4}{4!} y_n^{IV} + \ldots.. \quad \rightarrow (5)$$

8. Solve $y^1 = x - y^2$, $y(0) = 1$ using Taylor's series method and evaluate y(0.1), y(0.2) by step size h=0.1.

Sol: Given $y^1 = x - y^2$ →(1)

and $y(0) = 1$ →(2)

Here $x_0 = 0$, $y_0 = 1$.

Differentiating (1) w.r.t 'x', we get.

$y^{II} = 1 - 2yy^I$ →(3)

$y^{III} = -2(y.\, y^{II} + (y^I)^2)$ → (4)

$y^{IV} = -2[y.\, y^{III} + y^I.\, y^{II} + 2y^I.\, y^{II}]$ →(5)

$= -2(3y^I.\, y^{II} + y.\, y^{III})$ ……

Put $x_0 = 0$, $y_0 = 1$ in (1),(3),(4) and (5),

We get

$y_0^I = 0-1 = -1$,

$y_0^{II} = 1-2(1)(-1) = 3$,

$y_0^{III} = -2[(-1)^2) + (1)(3)] = -8$

y_0^{IV} = -2[3(-1) (3) + (1) (-8)] = -2(-9 -8) = 34.

Take h=0.1

Step1: By Taylor's series, we have

$$y_1 = y_0 + \frac{h}{1!}y_0^I + \frac{h^2}{2!}y_0^{II} + \frac{h^3}{3!}y_0^{III} + \frac{h^4}{4!}y_0^{IV} + \ldots.. \qquad \rightarrow(6)$$

on substituting the values of y_0, y_0^I, y_0^{II}, etc in equ (6) we get

$$y(0.1) = y_1 = 1 + \frac{0.1}{1}(-1) + \frac{(0.1)^2}{2}(3) + \frac{(0.1)^3}{6}(-8) + \frac{(0.1)^4}{24}(34) + \ldots.$$

$$= 1 - 0.1 + 0.015 - 0.00133 + 0.00014 + \ldots$$

$$= 0.91381$$

Step2: Let us find y(0.2), we start with (x_1, y_1) as the starting value.

Here $x_1 = x_0 + h = 0+0.1 = 0.1$ and $y_1 = 0.91381$

Put these values of x_1 and y_1 in (1),(3),(4) and (5),we get

$y_1^I = x_1 - y_1^2 = 0.1 - (0.91381)^2 = 0.1 - 0.8350487 = -0.735$

$y_1^{II} = 1 - 2y_1 \cdot y_1^I = 1- 2(0.91381)(-0.735) = 1 + 1.3433 = 2.3433$

$y_1^{III} = -2[(y_1^I)^2 + y_1 \cdot y_1^{II}] = -2[(-0.735)^2 + (0.91381)(2.3433)] = -5.363112$

$y_1^{IV} = -2[3.\, y_1^I\, y_1^{II} + y_1\, y_1^{III}]$

$= -2[3.(-0.735)(2.3433) + (0.91381)(-5.363112)]$

$= -2[(-5.16697) - 4.9] = 20.133953$

By Taylor's series expansion,

$$y_2 = y_1 + \frac{h}{1!}y_1^I + \frac{h^2}{2!}y_1^{II} + \frac{h^3}{3!}y_1^{III} + \frac{h^4}{4!}y_1^{IV} + \ldots.$$

$$\therefore y(0.2) = y_2 = 0.91381 + (0.1)(-0.735) + \frac{(0.1)^2}{2}(2.3433) + \frac{(0.1)^3}{6}(-5.363112) + \frac{(0.1)^4}{24}(20.133953) + \ldots.$$

$y(0.2) = 0.91381 - 0.0735 + 0.0117 - 0.00089 + 0.00008 = 0.8512$

9. Tabulate y(0.1), y(0.2) and y(0.3) using Taylor's series method given that $y^1 = y^2 + x$ and $y(0) = 1$

Sol: Given $y^I = y^2 + x$ →(1)

and $y(0) = 1$ →(2)

Here $x_0 = 0$, $y_0 = 1$.

Differentiating (1) w.r.t 'x', we get

$y^{II} = 2y \cdot y^I + 1$ →(3)

$y^{III} = 2[y \cdot y^{II} + (y^I)^2]$ →(4)

$y^{IV} = 2[y \cdot y^{III} + y^I y^{II} + 2 y^I y^{II}]$

$= 2[y \cdot y^{III} + 3 y^I y^{II}]$ →(5)

Put $x_0 = 0$, $y_0 = 1$ in (1), (3), (4) and (5), we get

$y_0^I = (1)^2 + 0 = 1$

$y_0^{II} = 2(1)(1) + 1 = 3,$

$y_0^{III} = 2((1)(3) + (1)^2) = 8$

$y_0^{IV} = 2[(1)(8) + 3(1)(3)]$

$= 34$

Take $h = 0.1$.

Step1: By Taylor's series expansion, we have

$$y(x_1) = y_1 = y_0 + \frac{h}{1!}y_0^I + \frac{h^2}{2!}y_0^{II} + \frac{h^3}{3!}y_0^{III} + \frac{h^4}{4!}y_0^{IV} + \ldots \quad \rightarrow (6)$$

on substituting the values of y_0, y_0^I, y_0^{II} etc in (6),we get

$$y(0.1) = y_1 = 1 + (0.1)(1) + \frac{(0.1)^2}{2}(3) + \frac{(0.1)^3}{6}(8) + \frac{(0.1)^4}{24}(34) + \ldots$$

$= 1 + 0.1 + 0.015 + 0.001333 + 0.000416$

$y_1 = 1.116749$

Step2: Let us find y(0.2),we start with (x_1, y_1) as the starting values

Here $x_1 = x_0 + h = 0 + 0.1 = 0.1$ and $y_1 = 1.116749$

Putting these values in (1),(3),(4) and (5), we get

$y_1^I = y_1^2 + x_1 = (1.116749)^2 + 0.1 = 1.3471283$

$y_1^{II} = 2y_1 y_1^I + 1 = 2(10116749)(1.3471283) + 1 = 4.0088$

$y_1^{III} = 2(y_1 y_1^{II} + (y_1^I)^2) = 2((1.116749)(4.0088) + (1.3471283)^2]$

= 12.5831

y_1^{IV} = 2y₁ y_1^{III} + 6 y_1^I y_1^{II} = 2(1.116749) (12.5831) + 6(1.3471283) (4.0088)

= 60.50653

By Taylor's expansion

$y(x_2) = y_2 = y_1 + \frac{h}{1!}y_1^I + \frac{h^2}{2!}y_1^{II} + \frac{h^3}{3!}y_1^{III} + \frac{h^4}{4!}y_1^{IV} + \ldots.$

$\therefore y(0.2) = y_2 = 1.116749 + (0.1)\ (1.3471283)$

$+ \frac{(0.1)^2}{2}\ (4.0088) + \frac{(0.1)^3}{6}\ (12.5831) + \frac{(0.1)^4}{24}\ (60.50653)$

$y_2 = 1.116749 + 0.13471283 + 0.020044 + 0.002097 + 0.000252$

= 1.27385

y(0.2) = 1.27385

Step3: Let us find y(0.3),we start with (x_2,y_2) as the starting value.

Here $x_2 = x_1 + h = 0.1 + 0.1 = 0.2$ and $y_2 = 1.27385$

Putting these values of x_2 and y_2 in eq (1), (3), (4) and (5),

we get

$y_2^I = y_2^2 + x_2 = (1.27385)^2 + 0.2 = 1.82269$

$y_2^{II} = 2y_2\, y_2^1 + 1 = 2(1.27385)\ (1.82269) + 1 = 5.64366$

$y_2^{III} = 2[y_2\, y_2^{II} + (y_2^I)^2] = 2[(1.27385)\ (5.64366) + (1.82269)^2]$

$= 14.37835 + 6.64439 = 21.02274$

$y_2^{IV} = 2y_2 + y_2^{III} + 6\, y_2^I \cdot y_1^{II}$

= 2(1.27385) (21.00274) + 6(1.82269)+(5.64366)

= 53.559635 + 61.719856 = 115.27949

By Taylor's expansion,

$y(x_3) = y_3 = y_2 + \frac{h}{1!}y_2^I + \frac{h^2}{2!}y_2^{II} + \frac{h^3}{3!}y_2^{III} + \frac{h^4}{4!}y_2^{IV} + \ldots..$

$y(0.3) = y_3 = 1.27385 + (0.1)\ (1.82269)$

$+ \frac{(0.1)^2}{2}(5.64366) + \frac{(0.1)^3}{6}(21.02274)$

$+ \frac{(0.1)^4}{24}(115.27949)$

= 1.27385 + 0.182269 + 0.02821

$+ 0.0035037 + 0.00048033 \quad = 1.48831$

$y(0.3) = 1.48831$

10. Solve $y^1 = x^2 - y$, $y(0) = 1$ using Taylor's series method and evaluate

$y(0.1), y(0.2), y(0.3)$ and $y(0.4)$ (correct to 4 decimal places)

Sol: Given $y^1 = x^2 - y$ $\rightarrow (1)$

and $y(0) = 1$ $\rightarrow (2)$

Here $x_0 = 0$, $y_0 = 1$

Differentiating (1) w.r.t 'x', we get

$y^{II} = 2x - y^1 \rightarrow (3)$

$y^{III} = 2 - y^{II} \rightarrow (4)$

$y^{IV} = -y^{III} \rightarrow (5)$

put $x_0 = 0$, $y_0 = 1$ in (1),(3),(4) and (5), we get

$y_0^I = x_0^2 - y_0 = 0-1 = -1,$

$y_0^{II} = 2x_0 - y_0^I = 2(0) - (-1) = 1$

$y_0^{III} = 2 - y_0^{II} = 2-1 = 1,$

$y_0^{IV} = - y_0^{III} = -1$ Take $h = 0.1$

Step1: by Taylor's series expansion

$$y(x_1) = y_1 = y_0 + \frac{h}{1!} y_0^I + \frac{h^2}{2!} y_0^{II} + \frac{h^3}{3!} y_0^{III} + \frac{h^4}{4!} y_0^{IV} + \ldots \rightarrow (6)$$

On substituting the values of y_0, y_0^I, y_0^{II} etc in (6), we get

$$y(0.1) = y_1 = 1 + (0.1)(-1) + \frac{(0.1)^2}{2}(1) + \frac{(0.1)^3}{6}(1) + \frac{(0.1)^4}{24}(-1) + \ldots$$

$= 1 - 0.1 + 0.005 + 0.01666 - 0.0000416$

$= 0.905125 \simeq 0.9051$ (4 decimal place).

Step2: Let us find $y(0.2)$ we start with (x_1, y_1) as the starting values

Here $x_1 = x_0 + h = 0 + 0.1 = 0.1$ and $y_1 = 0.905125$,

Putting these values of x_1 and y_1 in (1), (3), (4) and (5), we get

From 1,3,4,5 we get

$y_1^1 = x_1^2 - y_1 = (0.1)^2 – 0.905125 = -0.895125$

$y_1^{II} = 2x_1 - y_1^1 = 2(0.1) – (-0.895125) = 1.095125,$

$y_1^{III} = 2 - y_1^{II} = 2 – 1.095125 = 0.90475,$

$y_1^{IV} = - y_1^{III} = -0.904875,$

By Taylor's series expansion,

$$y(x_2) = y_2 = y_1 + \frac{h}{1!} y_1^I + \frac{h^2}{2!} y_1^{II} + \frac{h^3}{3!} y_1^{III} + \frac{h^4}{4!} y_1^{IV} + ...$$

$$y(0.2) = y_2 = 0.905125 + (0.1)(-0.895125) + \frac{(0.1)^2}{2} (1.09125)$$

$$+ \frac{(0.1)^3}{6} (1.095125) + \frac{(0.1)^4}{24} (-0.904875) + ...$$

$y(0.2) = y_2 = 0.905125 – 0.0895125 + 0.00547562 + 0.000150812$

$= 0.8212351 \simeq 0.8212$ (4 decimal places)

Step3: Let us find y(0.3), we start with (x_2,y_2) as the starting value

Here $x_2 = x_1 + h = 0.1+ 0.1 = 0.2$ and $y_2 = 0.8212351$

Putting these values of x_2 and y_2 in (1),(3),(4), and (5) we get

$y_2^1 = x_2^2 - y_2 = (0.2)^2 – 0.8212351= 0.04 – 0.8212351 = - 0.7812351$

$y_2^{II} = 2x_2 - y_2^1 = 2(0.2) + (0.7812351) = 1.1812351,$

$y_2^{III} = 2 - y_2^{II} = 2 – 1. 1812351 = 0.818765,$

$y_2^{IV} = - y_2^{III} = -0.818765,$

By Taylor's series expansion,

$$y(x_3) = y_3 = y_2 + \frac{h}{1!} y_2^I + \frac{h^2}{2!} y_2^{II} + \frac{h^3}{3!} y_2^{III} + \frac{h^4}{4!} y_2^{IV} + ...$$

$$y(0.3) = y_3 = 0. 8212351 + (0.1)(-0.7812351) + \frac{(0.1)^2}{2} (1.1812351)$$

$$+ \frac{(0.1)^3}{6} (0.818765) + \frac{(0.1)^4}{24} (-0.818765) + ...$$

$y(0.3) = y_3 = 0. 8212351– 0.07812351+ 0.005906 + 0.000136 –$

$= 0.749150 \simeq 0.7492$ (4 decimal places)

Step4: Let us find y(0.4), we start with (x_3,y_3) as the starting value

Here $x_3 = x_2 + h = 0.2 + 0.1 = 0.3$ and $y_3 = 0.749150$

Putting these values of x_3 and y_3 in (1),(3),(4), and (5) we get

$y_3^1 = x_3^2 - y_3 = (0.3)^2 - 0.749150 = -0.65915,$

$y_3^{II} = 2x_3 - y_3^1 = 2(0.3) + (0.65915) = 1.25915,$

$y_3^{III} = 2 - y_3^{II} = 2 - 1.25915 = 0.74085,$

$y_3^{IV} = - y_3^{III} = -0.74085,$

By Taylor's series expansion,

$$y(x_4) = y_4 = y_3 + \frac{h}{1!} y_3^I + \frac{h^2}{2!} y_3^{II} + \frac{h^3}{3!} y_3^{III} + \frac{h^4}{4!} y_3^{IV} + \ldots.$$

$$y(0.4) = y_4 = 0.749150 + (0.1)(-0.65915) + \frac{(0.1)^2}{2}(1.25915) + \frac{(0.1)^3}{6}(0.74085) + \frac{(0.1)^4}{24}(-0.74085) + \ldots.$$

$$y(0.4) = y_4 = 0.749150 - 0.065915 + 0.0062926 + 0.000123475 - 0.0000030$$

$$= 0.6896514 \simeq 0.6896 \text{ (4 decimal places)}$$

11. Solve $y^1 = x^2 - y$, $y(0) = 1$ using T.S.M and evaluate y(0.1),y(0.2),y(0.3) and y(0.4) (correct to 4 decimal place) Ans : 0.9051, 0.8212, 07492, 0.6896

12. Given the differentiating equation $y^1 = x^1 + y^2$, $y(0) = 1$. Obtain y(0.25) and y(0.5) by T.S.M.

    Ans: 1.3333, 1.81667

13. Solve $y^1 = xy^2 + y$, $y(0) = 1$ using Taylor's series method and evaluate y(0.1) and y(0.2)

    Ans: 1.111, 1.248.

3.6 Picard's Method

Consider the differential equation $\frac{dy}{dx} = f(x, y)$

Given that $y = y_0$ for $x = x_0$

Then $y^{(n)} = y_0 + \int_{x_0}^{x_n} f\left(x, y^{(n-1)}\right) dx, \quad n = 1, 2, 3, \cdots$

Problems

1. Find the value of y for x=0.4 by Picard's method, given that $\frac{dy}{dx} = x^2 + y^2,\ y(0) = 0.$

Sol: Given $\frac{dy}{dx} = x^2 + y^2,\ y(0) = 0$

By Picard's method $y^{(n)} = y_0 + \int_{x_0}^{x_n} f\left(x, y^{(n-1)}\right) dx, \quad n = 1, 2, 3, \cdots$

For the first approximation, replace y_0 by 0

$$y^{(1)} = 0 + \int_0^x (x^2 + 0)\, dx = \frac{x^3}{3}$$

Second approximation is $y^{(2)} = \int_0^x (x^2 + \left(\frac{x^3}{3}\right)^2)\, dx = \int_0^x (x^2 + \frac{x^6}{9})\, dx = \frac{x^3}{3} + \frac{x^7}{63}$

Calculation of $y^{(3)}$ is tedious and hence approximate value is $y^{(2)}$

For x=0.4, $y = \frac{(0.4)^3}{3} + \frac{(0.4)^7}{63} = 0.02133 + 0.00026 = 0.0214$

2. Solve Find the value of y at x=0.1 by Picard's method, given that

$\frac{dy}{dx} = \frac{y-x}{y+x}, y(0) = 1$

Sol: Given $\frac{dy}{dx} = \frac{y-x}{y+x}, y(0) = 1$

By Picard's method $y^{(n)} = y_0 + \int_{x_0}^{x_n} f\left(x, y^{(n-1)}\right) dx = y_0 + \int_0^x \frac{y-x}{y+x}\, dx$

For the first approximation, replace y_0 by 1

$$y^{(1)} = 1 + \int_0^x \frac{1-x}{1+x}\, dx = 1 + \int_0^x -1 + \frac{2}{1+x}\, dx$$

$$y^{(1)} = 1 + [-x + 2\log(1+x)]_0^x$$

$$y^{(1)} = 1 - x + 2\log(1+x)$$

Second approximation is $y^{(2)} = 1 + \int_0^x \frac{1-x+2\log(1+x)-x}{1-x+2\log(1+x)+x}\, dx$

Which is very difficult to integrate

Hence we use the first approximation itself a s the value of y

$$\therefore y(x) = y^{(1)} = 1 - x + 2\log(1+x)$$

Put x=0.1, we get

$$y(0.1) = 1 - 0.1 + 2\log(1 + 0.1) = 1.0906$$

3.6 EULER'S METHOD

It is the simplest one-step method and it is less accurate. Hence it has a limited application.

Consider the differential equation $\frac{dy}{dx} = f(x,y)$ → (1)

With $y(x_0) = y_0$ → (2)

Consider the first two terms of the Taylor's expansion of $y(x)$ at $x = x_0$

$$y(x) = y(x_0) + (x - x_0)\, y^1(x_0) \quad \rightarrow (3)$$

from equation (1) $y^1(x_0) = f(x_0, y(x_0)) = f(x_0, y_0)$

Substituting in equation (3)

$\therefore y(x) = y(x_0) + (x - x_0)\, f(x_0,y_0)$

At $x = x_1$, $y(x_1) = y(x_0) + (x_1 - x_0)\, f(x_0,y_0)$

$\therefore y_1 = y_0 + h\, f(x_0,y_0)$ where $h = x_1 - x_0$

Similarly at $x = x_2$, $y_2 = y_1 + h\, f(x_1,y_1)$,

Proceeding as above, $\mathbf{y_{n+1} = y_n + h\, f(x_n,y_n)}$

This is known as Euler's Method

1. **Using Euler's method solve for x = 2 from $\frac{dy}{dx} = 3x^2 + 1, y(1) = 2$, by taking step size**
(I) h = 0.5 and (II) h=0.25

Sol: Here $\frac{dy}{dx} = f(x,y) = 3x^2 + 1$, $x_0 = 1$, $y_0 = 2$

Euler's algorithm is $y_{n+1} = y_n + h\ f(x_n,y_n)$, n = 0,1,2,3,..... → (1)

(I) $h = 0.5$ $\therefore x_1 = x_0 + h = 1+0.5 = 1.5$

Taking n = 0 in (1) , we have $x_2 = x_1 + h = 1.5 + 0.5 = 2$

$y_1 = y_0 + h\, f(x_0,y_0)$

i.e. $y_1 = y(1.5) = 2 + (0.5)\, f(1,2) = 2 + (0.5)\,(3 + 1) = 2 + (0.5)(4)=4$

Here $x_1 = x_0 + h = 1 + 0.5 = 1.5$

$\therefore y(1.5) = 4 = y_1$

Taking n = 1 in (1),we have

$y_2 = y_1 + h\, f(x_1,y_1)$

i.e. $y(x_2) = y_2 = 4 + (0.5)\, f(1.5,4) = 4 + (0.5)[3(1.5)^2 + 1] = 7.875$

Here $x_2 = x_4 + h = 1.5 + 0.5 = 2$

$\therefore y(2) = 7.875$

(II) $h = 0.25$ $x_0 = 1, y_0 = 2$ $\therefore x_1 = 1.25,\ x_2 = 1.50,\ x_3 = 1.75,\ x_4 = 2$

Taking n = 0 in (1), we have

$y_1 = y_0 + h\, f(x_0,y_0)$

i.e. $y(x_1) = y_1 = 2 + (0.25)\, f(1,2) = 2 + (0.25)\ (3 + 1) = 3 = y(1.25)$

$y(x_2) = y_2 = y_1 + h\, f(x_1,y_1)$

i.e. $y(x_2) = y_2 = 3 + (0.25)\, f(1.25,3)$

$= 3 + (0.25)[3(1.25)^2 + 1]$

$= 4.42188$

Here $x_2 = x_1 + h = 1.25 + 0.25 = 1.5$

$\therefore y_2 = y(1.5) = 4.42188$

Taking n = 2 in (1), we have

i.e. $y(x_3) = y_3 = y_2 + h\, f(x_2,y_2)$

$= 4.42188 + (0.25)\, f(1.5,2)$

$= 4.42188 + (0.25)\ [3(1.5)^2 + 1]$

$= 6.35938$

Here $x_3 = x_2 + h = 1.5 + 0.25 = 1.75$

$\therefore y(1.75) = 6.\ 35938 = y_3$

Taking n = 4 in (1), we have

$y(x_4) = y_4 = y_3 + h\, f(x_3,y_3)$

i.e. $y(x_4) = y_4 = 6.35938 + (0.25)\, f(1.75,2)$

$= 6.35938 + (0.25)[3(1.75)^2 + 1]$

$y(x_4) = 8.90626 = y(2)$

Note that the difference in values of y(2) in both cases (i.e. when h = 0.5 and when h = 0.25).The accuracy is improved significantly when h is reduced to 0.25 (Example significantly of the eqn is $y = x^3 + x$ and with this $y(2) = y_2 = 10$

2. **Solve by Euler's method, $y^1 = x + y$, $y(0) = 1$ and find y(0.3) taking step size h = 0.1. compare the result obtained by this method with the result obtained by analytical solution**

Sol: Given D.E. is $y^1 = f(x,y) = x + y$, $y(0) = 1, h=0.1, x_0=0, x_1=0.1, x_2=0.2, x_3=0.3, y_0=1$

From Euler's method

$y_1 = y_0 + h\, f(x_0,y_0) = 1+0.1(0+1) = 1.1$

$y(x_2) = y_2 = y_1 + h\, f(x_1,y_1) = 1.1+0.1(0.1+1.1) = 1.22$

$y(x_3) = y_3 = y_2 + h\, f(x_2,y_2) = 1.22+0.1(0.2+1.22) = 1.362$

$y_1 = 1.1 = y(0.1)$,

$y_2 = y(0.2) = 1.22$

$y_3 = y(0.3) = 1.362$

Analytical method (linear d.e. method)

$y^1 = x + y$

dy/dx-y=x

P=-1,Q=x

I.F.=e^int(Pdx)=e^int(-dx)=e^-x

Sol. Y(I.F.)=int(Q(i.f.)dx+c

Y(e^-x)=int(xe^-x)dx+c=e^-x(-x-1)+c

Divide by e^-x on b.s.

Solution y=-x-1+ce^x

Put x=0,y=1 then

1=-0-1+c

C=2

General solution y=-x-1+2e^x

Particular solution is $y(x) = 2e^x – (x + 1)$

Hence analytical values y(0.1) = 1.11034, y(0.2) = 1.3428, y(0.3) = 1.5997

We shall tabulate the result as follows

X	0	X_1=0.1	X_2=0.2	X_3=0.3
Euler y(numerical)	1	1.1	1.22	1.362
Linear y(analytical)	1	1.11034	1.3428	1.3997

The value of y deviate from the execute value as x increases. This indicate that the method is not accurate

3. **Solve by Euler's method $y^1 + y = 0$ given y(0) = 1 and find y(0.04) taking step size**

   h = 0.01 Ans: 0.9606

4. **Using Euler's method, solve y at x = 0.1 from y^1 = x+ y +xy, y()) = 1 taking step size h = 0.025.**

5. **Given that $\frac{dy}{dx}$ = xy ,y(0) = 1 determine y(0.1),using Euler's method.** h =0.1

Sol: The given differentiating equation is $\frac{dy}{dx}$ = xy, y(0) = 1 ,a = 0,b=0.1

Here f(x,y) = xy , x_0 = 0 and y_0 = 1

Since h is not given much better accuracy is obtained by breaking up the interval (0,0.1) in to five steps.

i.e. h = $\frac{b-a}{5}$ = $\frac{0.1}{5}$ = 0.02

Euler's algorithm is $y_{n+1} = y_n + h\, f(x_n,y_n)$ →(1)

∴ From (1) form = 0, we have

$y_1 = y_0 + h\, f(x0,y_0)$

= 1 + (0.02) f(0,1)

= 1 + (0.02) (0)

= 1

Next we have $x_1 = x_0 + h = 0 + 0.02 = 0.02$

∴ From (1), form = 1,we have

$y_2 = y_1 + h\, f(x_1,y_1)$

= 1 + (0.02) f(0.02,1)

= 1 + (0.02) (0.02)

= 1.0004

Next we have $x_2 = x_1 + h = 0.02 + 0.02 = 0.04$

∴ From (1), form = 2,we have

$y_3 = y_2 + h\, f(x_2,y_2)$

= 1.004 + (0.02) (0.04) (1.0004)

= 1.0012

Next we have $x_3 = x_2 + h = 0.04 + 0.02 = 0.06$

∴ From (1), form = 3,we have

$y_4 = y_3 + h\, f(x_3,y_3)$

= 1.0012 + (0.02) (0.06) (1.00012)

= 1.0024.

Next we have $x_4 = x_3 + h = 0.06 + 0.02 = 0.08$

∴ From (1), form = 4,we have

$y_5 = y_4 + h\ f(x_4,y_4)$

$= 1.0024 + (0.02)\ (0.08)\ (1.00024)$

$= 1.0040.$

Next we have $x_5 = x_4 + h = 0.08 + 0.02 = 0.1$

When $x = x_5$, $y \simeq y_5$

$\therefore y = 1.0040$ when $x = 0.1$

6. Solve by Euler's method $y^1 = \frac{2y}{x}$ given y(1) = 2 and find y(2).

7. Given that $\frac{dy}{dx} = 3x^2 + y$, y(0) = 4. Find y(0.25) and y(0.5) using Euler's method

Sol: given $\frac{dy}{dx} = 3x^2 + y$ and $y(0) = 4$.

Here $f(x,y) = 3x^2 + y$, $x_0 = 0$, $y_0 = 4$

Consider $h = 0.25$

Euler's algorithm is $y_{n+1} = y_n + h\ f(x_n,y_n)$ →(1)

$\therefore$ From (1), for n = 0, we have

$y_1 = y_0 + h\ f(x_0,y_0)$

$= 2 + (0.25)[0 + 4]$

$= 2 + 1$

$= 3$

Next we have $x_1 = x_0 + h = 0 + 0.25 = 0.25$

When $x = x_1$, $y_1 \simeq y$

$\therefore y_1 = 3$ when $x_1 = 0.25$

$\therefore$ From (1), for n = 1, we have

$y_2 = y_1 + h\ f(x_1,y_1)$

$= 3 + (0.25)[3.(0.25)^2 + 3]$

$= 3.7968$

Next we have $x_2 = x_1 + h = 0.25 + 0.25 = 0.5$

When $x = x_2$, $y \simeq y_2$

$\therefore y = 3.7968$ when $x = 0.5$.

8. **Solve first order differential equation $\frac{dy}{dx}$ = $\frac{y-x}{y+x}$, y(0) = 1 and estimate y(0.1) using Euler's method (5 steps).h=0.02**

   Ans: 1.0928

9. **Use Euler's method to find approximate value of solution of $\frac{dy}{dx}$ = y-x + 5 at x = 2-1 and 2-2with initial contention y(0.2) = 1**

3.7 Modified Euler's method

It is given by $y^{(i)}{}_{k+1} = y_k + h/2 f\left[(x_k, y_k) + f(x_{k+1}, 1)_{k+1}{}^{(i-1)}\right], i = 1, 2....., ki = 0, 1.....$

Working rule :

i)Modified Euler's method

$$y^{(i)}{}_{k+1} = y_k + h/2 f\left[(x_k, y_k) + f(x_{k+1}, 1)_{k+1}{}^{(i-1)}\right], i = 1, 2....., ki = 0, 1.....$$

ii) When $i = 1$ $y^{0}{}_{k+1}$ can be calculated from Euler's method

iii) K=0, 1……… gives number of iteration. $i = 1, 2...$

gives number of times, a particular iteration k is repeated

Suppose consider dy/dx=f(x, y) -------- (1) with $y(x_0) = y_0$----------- (2)

To find $y(x_1) = y_1$ at $x = x_1 = x_0 + h$

Now take k=0 in modified Euler's method

We get $y_1^{(1)} = y_0 + h/2\left[f(x_0, y_0) + f\left(x_1, y_1^{(i-1)}\right)\right]$.......................... (3)

Taking i=1, 2, 3...k+1 in eqn (3), we get

$y_1^{(0)} = y_0 + h/2\left[f(x_0, y_0)\right]$ (By Euler's method)

$$y_1^{(1)} = y_0 + h/2\left[f(x_0, y_0) + f\left(x_1, y_1^{(0)}\right)\right]$$

$$y_1^{(2)} = y_0 + h/2\left[f(x_0, y_0) + f\left(x_1, y_1^{(1)}\right)\right]$$

$$y_1^{(k+1)} = y_0 + h/2\left[f(x_0, y_0) + f\left(x_1, y_1^{(k)}\right)\right]$$

If two successive values of $y_1^{(k)}, y_1^{(k+1)}$ are sufficiently close to one another, we will take the common value as $y_2 = y(x_2) = y(x_1 + h)$

We use the above procedure again

1) using modified Euler's method find the approximate value of x when $x=0.3$ given that $dy/dx = x+y$ and $y(0)=1$

sol: Given $dy/dx = x+y$ and $y(0)=1$

Here $f(x,y)=x+y, x_0=0,$ and $y_0=1$

Take h = 0.1 which is sufficiently small

Here $x_0=0, x_1=x_0+h=0.1, x_2=x_1+h=0.2, x_3=x_2+h=0.3$

The formula for modified Euler's method is given by

$$y_{k+1}^{(i)} = y_k + h/2\left[f(x_k+y_k)+f\left(x_{k+1}, y_{k+1}^{(i-1)}\right)\right] \to (1)$$

Step1: To find $y_1 = y(x_1) = y(0.1)$

Taking k = 0 in eqn(1)

$$y_{k+1}^{(i)} = y_0 + h/2\left[f(x_0+y_0)+f\left(x_1, y_1^{(i-1)}\right)\right] \to (2)$$

when i = 1 in eqn (2)

$$y_1^{(i)} = y_0 + h/2\left[f(x_0, y_0)+f\left(x_1, y_1^{(0)}\right)\right]$$

First apply Euler's method to calculate $y_1^{(0)}$ = y_1

$$\therefore\ y_1^{(0)} = y_0 + h\, f(x_0, y_0)$$

= 1+(0.1)f(0.1)

= 1+(0.1)

= 1.10

$$now\left[x_0=0, y_0=1, x_1=0.1, y_1(0)=1.10\right]$$

$$\therefore\ y_1^{(1)} = y_0 + 0.1/2\left[f(x_0, y_0)+f\left(x_1, y_1^{(0)}\right)\right]$$

= 1+0.1/2[f(0,1) + f(0.1,1.10)

= 1+0.1/2[(0+1)+(0.1+1.10)]

= 1.11

When i=2 in eqn (2)

$$y_1^{(2)} = y_0 + h/2\left[f(x_0, y_0)+f\left(x_1, y_1^{(1)}\right)\right]$$

= 1+0.1/2[f(0.1)+f(0.1,1.11)]

= 1 + 0.1/2[(0+1)+(0.1+1.11)]

= 1.1105

$$y_1^{(3)} = y_0 + h/2\left[f\left(x_0, y_0\right) + f\left(x_1, y_1^{(2)}\right)\right]$$

= 1+0.1/2[f(0,1)+f(0.1 , 1.1105)]

= 1+0.1/2[(0+1)+(0.1+1.1105)]

= 1.1105

Since $y_1^{(2)} = y_1^{(3)}$

∴ y_1 = 1.1105

Step:2 To find $y_2 = y(x_2) = y(0.2)$

Taking k = 1 in eqn (1) , we get

$$y_2^{(i)} = y_1 + h/2\left[f\left(x_1, y_1\right) + f\left(x_2, y_2^{(i-1)}\right)\right] \rightarrow (3) \qquad i = 1,2,3,4,.....$$

For i = 1

$$y_2^{(1)} = y_1 + h/2\left[f\left(x_1, y_1\right) + f\left(x_2, y_2^{(0)}\right)\right]$$

$y_2^{(0)}$ is to be calculate from Euler's method

$$y_2^{(0)} = y_1 + h\ f\left(x_1, y_1\right)$$

= 1.1105 + (0.1) f(0.1 , 1.1105)

= 1.1105+(0.1)[0.1+1.1105]

= 1.2316

$$\therefore y_2^{(1)} = 1.1105 + 0.1/2\left[f\left(0.1, 1.1105\right) + f\left(0.2, 1.2316\right)\right]$$

= 1.1105 +0.1/2[0.1+1.1105+0.2+1.2316]

= 1.2426

$$y_2^{(2)} = y_1 + h/2\left[f\left(x_1, y_1\right) + f\left(x_2 y_2^{(1)}\right)\right]$$

= 1.1105 + 0.1/2[f(0.1 , 1.1105) , f(0.2 . 1.2426)]

= 1.1105 + 0.1/2[1.2105 + 1.4426]

$= 1.1105 + 0.1(1.3266)$

$= 1.2432$

$$y_2^{(3)} = y_1 + h/2\left[f(x_1, y_1) + f\left(x_2 y_2^{(2)}\right)\right]$$

= 1.1105+0.1/2[f(0.1,1.1105)+f(0.2 , 1.2432)]

= 1.1105+0.1/2[1.2105+1.4432)]

= 1.1105 + 0.1(1.3268)

= 1.2432

Since $y_2^{(3)} = y_2^{(3)}$

Hence $y_2 = 1.2432$

Step:3

To find $y_3 = y(x_3) = y\ y(0.3)$

Taking k =2 in eqn (1) we get

$$y_3^{(1)} = y_2 + h/2\left[f(x_2, y_2) + f\left(x_3, y_3^{(i-1)}\right)\right] \to (4)$$

For i = 1 ,

$$y_3^{(1)} = y_2 + h/2\left[f(x_2, y_2) + f\left(x_3, y_3^{(0)}\right)\right]$$

$y_3^{(0)}$ is to be evaluated from Euler's method .

$$y_3^{(0)} = y_2 + h\, f(x_2, y_2)$$

= 1.2432 +(0.1) f(0.2 , 1.2432)

= 1.2432+(0.1)(1.4432)

= 1.3875

$\therefore\ y_3^{(1)}$ = 1.2432+0.1/2[f(0.2 , 1.2432)+f(0.3, 1.3875)]

= 1.2432 + 0.1/2[1.4432+1.6875]

= 1.2432+0.1(1.5654)

= 1.3997

$$y_3^{(2)} = y_2 + h/2\left[\ f(x_2, y_2) + f\left(x_3, y_3^{(1)}\right)\right]$$

= 1.2432+0.1/2[1.4432+(0.3+1.3997)]

= 1.2432+ (0.1) (1.575)

= 1.4003

$$y_3^{(3)} = y_2 + h/2\left[\ f(x_2, y_2) + f\left(x_3, y_3^{(2)}\right)\right]$$

= 1.2432+0.1/2[f(0.2 , 1.2432)+f(0.3 , 1.4003)]

= 1.2432 + 0.1(1.5718)

= 1.4004

$$y_3^{(4)} = y_2 + h/2\left[\ f(x_2, y_2) + f\left(x_3, y_3^{(3)}\right)\right]$$

= 1.2432 + 0.1/2[1.4432+1.7004]

= 1.2432+(0.1)(1.5718)

= 1.4004

Since $y_3^{(3)} = y_3^{(4)}$

Hence $y_3 = 1.4004$ $\therefore$ The value of y at x = 0.3 is 1.4004

2 . Find the solution of $\frac{dy}{dx}$ = x-y , y(0)=1 at x =0.1 , 0.2 ,0.3 , 0.4 and 0.5 . Using modified Euler's method

Sol . Given $\frac{dy}{dx}$ = x-y and y(0) = 1

Here f(x,y) = x-y , x_0 = 0 and y_0 = 1

Consider h = 0.1 so that

x = 0.1 , x_2 = 0.2 , x_3 =0.3 , x_4 = 0.4 and x_5 = 0.5

The formula for modified Euler's method is given by

$$y_{k+1}^{(i)} = y_k + h/2 \; f(x_k, y_k) + f x_{k+1}, y_{k+1}^{(i-1)} \rightarrow (1)$$

Where k = 0,1, 2, 3,….. i = 1, 2, 3,…..

x	$f(x_k, y_k) = x_k - y_k$	$\frac{1}{2}\left[f(x_k, y_k) + f\left(x_{k+1}, y_{k+1}^{(i-1)}\right)\right]$	$y_{k+1}^{(i)} = y_k + h/2\left[f(x_k, y_k) + f\left(x_{k+1}, y_{k+1}^{(i-1)}\right)\right]$
K = 0			
0.0	0-1=-1	-	1+(0.1)(-1)=0.9 = $y_1^{(0)}$

0.1(i=1)	0-1=-1	½(-1-0.8) = -0.9	1+(0.1)(-0.9)=0.91
0.1(i=2)	0-1=-1	½(-1-0.81)= -0.905	1+(0.1)(-0.905)=0.9095
0.1(i=3)	0-1=-1	½(-1-0.80.95)= -0.90475	1+(0.1)(-0.90475)=0.9095
K=1			
0.1	0.1-0.9095= -0.8095	-	0.9095+(0.1)(-0.8095)=0.82855
0.2(i=1)	-0.8095	½(-0.8095-0.62855)	0.9095+(0.1)(-0.719025)=0.8376
0.2(i=2)	-0.8095	½(-0.8095-0.6376)	0.9095+(0.1)(-0.72355)=0.8371
0.2(i=3)	-0.8095	½(-0.8095-0.6371)	0.9095+(0.1)(-0.7233)=0.8372
0.2(i=4)	-0.8095	½(-0.8095-0.6372)	0.9095+(0.1)(-0.72355)=0.8371
K=2			
0.2	0.2-0.8371=-0.6371	-	0.8371+(0.1)(-0.6371)=0.7734

0.3(i=1)	= -0.6371	½(-0.6371-0.4734)	0.8371+(0.1)(-0.555)=0.7816
0.3(i=2)	= -0.6371	½(-0.6371-0.4816)	0.8371-0.056=0.7811
0.3(i=3)	= -0.6371	½(-0.6371-0.4811)	0.8371-0.05591=0.7812
0.3(i=4)	= -0.6371	½(-0.6371-0.4812)	0.8371-0.055915 = 0.7812
K =3			
0.3(i=1)	0.3-0.7812	-	0.7812+(0.1)(-0.4812) = 0.7331
0.4(i=1)	-0.4812	½(-0.4812-0.4311)	0.7812-0.0457 = 0.7355
0.4(i=2)	-0.4812	½(-0.4812-0.4355)	0.7812-0.0458 = 0.7354
0.4(i=3)	-0.4812	½(-0.4812-0.4354)	0.7812-0.0458 = 0.7354
K=4			
0.4	-0.3354	-	0.7354-0.03354 = 0.70186
0.5	-0.3354	½(-0.3354-0.301816)	0.7354-0.03186 = 0.7035
0.5	-0.3354	½(-0.3354-0.30354)	0.7354-0.0319 = 0.7035

3. Find y(0.1) and y(0.2) using modified Euler's formula given that $dy/dx=x^2-y, y(0)=1$

[consider $h=0.1, y_1=0.90523, y_2=0.8214$]

4. Given $dy/dx=-xy^2, y(0)=2$ compute y(0.2) in steps of 0.1

Using modified Euler's method

[h=0.1, y_1=1.9804, y_2=1.9238]

5. Given $y^1 = x+\sin y$, y(0)=1 compute y(0.2) and y(0.4) with h=0.2 using modified Euler's

method

[y_1=1.2046, y_2=1.4644]

3.8 Runge – Kutta Methods

I. Second order R-K Formula

$y_{i+1} = y_i+1/2\ (K_1+K_2)$,

Where $K_1 = h\ (x_i, y_i)$

$K_2 = h\ (x_i+h, y_i+k_1)$

For i= 0,1,2-------

II. Third order R-K Formula

$y_{i+1} = y_i+1/6\ (K_1+4K_2+ K_3)$,

Where $K_1 = h\ (x_i, y_i)$

$K_2 = h\ (x_i+h/2, y_0+k_1/2)$

$K_3 = h\ (x_i+h, y_i+2k_2-k_1)$

For i= 0,1,2-------

III. Fourth order R-K Formula

$y_{i+1} = y_i+1/6\ (K_1+2K_2+ 2K_3+K_4)$,

Where $K_1 = h\ (x_i, y_i)$

$K_2 = h\ (x_i+h/2, y_i+k_1/2)$

$K_3 = h\ (x_i+h/2, y_i+k_2/2)$

$K_4 = h\ (x_i+h, y_i+k_3)$

For i= 0,1,2-------

1. Using Runge-Kutta method of second order, find $y(2.5)$ from $\frac{dy}{dx}$ = $\frac{x+y}{x}$, y(2)=2 , h = 0.25 .

Sol: Given $\frac{dy}{dx}$ = $\frac{x+y}{x}$, y(2) = 2 .

Here f(x, y) = $\frac{x+y}{x}$, $x_0 = 0$, $y_0=2$ and h = 0.25

$\therefore$ $x_1 = x_0+h = 2+0.25 = 2.25$, $x_2 = x_1+h = 2.25+0.25 = 2.5$

By R-K method of second order,

$$y_{i+1} = y_i + 1/2(k_1 + k_2), k_1 - hf(x_i + h, y_i + k_1), i = 0,1.... \rightarrow (1)$$

Step -1:-

To find $y(x_1)$i.e y(2.25) by second order R - K method taking i=0 in eqn(i)

We have $y_1 = y_0 + \frac{1}{2}(k_1 + k_2)$

Where k_1= hf (x_0,y_0), k_2= hf (x_0+h,y_0+k_1)

f (x_0,y_0)=f(2,2)=2+2/2=2

k_1=hf (x_0,y_0)=0.25(2)=0.5

k_2= hf (x_0+h,y_0+k_1)=(0.25)f(2.25,2.5)

=(0.25)(2.25+2.5/2.25)=0.528

$\therefore$ y_1=y(2.25)=2+1/2(0.5+0.528)

=2.514

Step2:

To find $y(x_2)$ i.e., y(2.5)

i=1 in (1)

x_1=2.25,y_1=2.514,and h=0.25

$y_2=y_1+1/2(k_1+k_2)$

where $k_1=h\,f((x_1,y_1)=(0.25)f(2.25,2.514)$

$=(0.25)[2.25+2.514/2.25]=0.5293$

$$k_2 = h\,f\left(x_0+h, y_0+k_1\right) = (0.1)\,f\left(0.1, 1-0.1\right) = (0.1)(-0.9) = -0.09$$

$=(0.25)[2.5+2.514+0.5293/2.5]$

$=0.55433$

$y_2 = y\ (2.5)=2.514+1/2(0.5293+0.55433)$

$=3.0558$

$\therefore y=3.0558$ when $x = 2.5$

Obtain the values of y at x=0.1,0.2 using R-K method of

(i)second order (ii)third order (iii)fourth order for the diff eqn $y^1+y=0, y(0)=1$

Sol: Given $dy/dx = -y$, $y(0)=1$

$f(x,y) = -y$, $x_0 = 0$, $y_0 = 1$

Here $f(x,y) = -y$, $x_0 = 0$, $y_0 = 1$ take $h = 0.1$

$\therefore x_1 = x_0+h = 0.1,$

$x_2 = x_1+h = 0.2$

Second order:

step1: To find $y(x_1)$ i.e $y(0.1)$ or y_1

by second-order R-K method,we have

$y_1 = y_0+1/2(k_1+k_2)$

where $k_1=hf(x_0,y_0)=(0.1)\,f(0,1) = (0.1)(-1)= -0.1$

$k_2= hf(x_0+h, y_0+k_1)= (0.1)\,f(0.1, 1-0.1) = (0.1)(-0.9) = -0.09$

$y_1=y(0.1)=1+1/2(-0.1-0.09)=1-0.095=0.905$

$\therefore$ y =0.905 when x=0.1

Step2:

To find y_2 i.e $y(x_2)$ i.e y(0.2)

Here x_1 = 0.1, y_1 = 0.905 and h=0.1

By second-order R-K method, we have

$y_2 = y(x_2) = y_1 + 1/2(k_1 + k_2)$

Where $k_1 = h\ f(x_1, y_1)$ =(0.1)f(0.1,0.905)=(0.1)(-0.905)=-0.0905

$$k_2 = h\ f(x_1 + h, y_1 + k_1) = (0.1)\ f(0.2, 0.905 - 0.0905)$$
$$= (0.1)\ f(0.2, 0.8145) = (0.1)(-0.8145)$$
$$= -0.08145$$

y_2 = y(0.2)=0.905+1/2(-0.0905-0.08145)

= 0.905- 0.085975 = 0819025

Third order

Step1:

To find y_1 i.e $y(x_1)$= y(0.1)

By Third order Runge kutta method

$$y_1 = y_0 + 1/6(k_1 + 4k_2 + k_3)$$

where $k_1 = h\ f(x_0, y_0)$ = (0.1) f (0.1) = (0.1) (-1) = -0.1

$$k_2 = h\ f(x_0 + h/2, y_0 + k_1/2) = (0.1)\ f(0.1/2, 1 - 0.1/2) = (0.1)\ f(0.05, 0.95)$$
$$= (0.1)(-0.95) = -0.095$$

and $k_3 = h\ f((x_0 + h, y_0 + 2k_2 - k_1)$

(0.1) f (0.1,1+2(-0.095)+0.1)= -0.905

Hence y_1 = 1+1/6(-0.1+4(-0.095)-0.09) = 1+1/6 (-0.57) = 0.905

y_1=0.905 i.e y(0.1)= 0.905

Step2:

To find y_2,i.e $y(x_2)= y(0.2)$

Here $x_1=0.1, y_1=0.905$ and $h = 0.1$

Again by 2nd order R-K method

$y_2 = y_1+1/6(k_1+4k_2+k_3)$

Where $k_1=h\ f(x_1, y_1) = (0.1)f\ (0.1,0.905)= -0.0905$

$k_2 = h\ f\ (x_1+h/2,y_1+k_1/2)=(0.1)f(0.1+0.2,0.905 - 0.0905)= -(0.1)\ f\ (0.15, 0.85975)= (0.1)\ (-0.85975)$

and $k_3 = h\ f((x_1+h,y_1+2k_2-k_1)=(0.1)f(0.2,0.905+2(0.08975)+0.0905= -0.082355$

hence $y_2 = 0.905+1/6(-0.0905+4(-0.085975)-0.082355)=0.818874$

$\therefore y = 0.905$ when $x = 0.1$

And $y =0.818874$ when $x =0.2$

fourth order:

step1:

$x_0=0,y_0=1,h=0.1$ To find y_1 i.e $y(x_1)=y(0.1)$

By 4th order R-K method, we have

$y_1 = y_0+1/6(k_1+2k_2+2k_3+k_4)$

Where $k_1=h\ f(x_0,y_0)=(0.1)f(0.1)= -0.1$

$k_2= h\ f\ (x_0+h/2,\ y_0+k1/2) = -0.095$

and $k_3= h\ f((x_0+h/2,y_0+k_2/2)=(0.1)f\ (0.1/2,1-0.095/2)$

$= (0.1)f(0.05,0.9525)$

$= -0.09525$

and $k_4= h\ f(x_0+h,y_0+k_3)$

$= (0.1)\ f(0.1,1-0.09525)=(0.1)f(0.1,0.90475)$

$=-0.090475$

Hence $y_1=1+1/6(-0.1)+2(-0.095)+2(0.09525)-0.090475)$

$=1+1/6(-0.570975)+1-0.951625 = 0.9048375$

Step2:

To find $y_2, i.e., y(x_2) = y(0.2), y_1 = 0.9048375, i.e., y(0.1) = 0.9048375$

Here $x_1 = 0.1$, $y_1=0.9048375$ and $h = 0.1$

Again by 4th order R-K method, we have

$y_2 = y_1+1/6(k_1+2k_2+2k_3+k_4)$

Where $k_1=h\,f(x_1,y_1)=(0.1)f(0.1,0.9048375)=-0.09048375$

$k_2= hf\,(x_1+h/2,y_1+k_1/2)=(0.1)f(0.1+0.1/2,0.9048375 -0.09048375 /2)=-0.08595956$

and $k_3=hf(x_1+h/2, y_1+k_2/2)=(0.1)f(0.15,0.\ 8618577)= -0.08618577$

$k_4 =h\,f(x_1+h,y_1+k_3)=(0.1)f(0.2,0.86517)$

$= -0.08186517$

Hence $y_2 = 0.09048375+1/6(-0.09048375-2(0.08595956)-2(0.08618577)-0.08186517$

$=0.9048375-0.0861065$

$=0.818731$

y = 0.9048375 when x =0.1 and y =0.818731

3. Apply the 4th order R-K method to find an approximate value of y when x=1.2 in steps of 0.1,given that $y^1 = x^2+y^2$,y (1)=1.5

sol. Given $y^1= x^2+y^2$,and $y(1)=1.5$

Here $f(x,y)= x^2+y^2$, $y_0 =1.5$ and $x_0=1$,h=0.1

So that $x_1=1.1$ and $x_2=1.2$

Step1:

To find y_1 i.e., $y(x_1)$

by 4th order R-K method we have

$y_1=y_0+1/6\ (k_1+2k_2+2k_3+k_4)$

$k_1=hf(x_0,y_0)=(0.1)f(1,1.5)=(0.1)\ [1^2+(1.5)^2]=0.325$

$k_2=hf\ (x_0+h/2,y_0+k_1/2)=(0.1)f(1+0.05,1.5+0.325)=0.3866$

and $k_3=hf((x_0+h/2,y_0+k_2/2)=(0.1)f(1.05,1.5+0.3866/2)=(0.1)[(1.05)^2+(1.6933)^2]$

$=0.39698$

$k_4=hf(x_0+h,y_0+k_3)=(0.1)f(1.0,1.89698)$

$=0.48085$

Hence

$$y_1=1.5+\frac{1}{6}\left[0.325+2(0.3866)+2(0.39698)+0.48085\right]$$
$$=1.8955$$

Step2:

To find y_2, i.e., $y(x_2)=y(1.2)$

Here $x_1=0.1,y_1=1.8955$ and $h=0.1$

by 4^{th} order R-K method we have

$y_2=y_1+1/6(k_1+2k_2+2k_3+k_4)$

$k_1=hf(x_1,y_1)=(0.1)f(0.1,1.8955)=(0.1)\ [1^2+(1.8955)^2]=0.48029$

$k_2=hf\ (x_1+h/2,y_1+k_1/2)=(0.1)f(1.1+0.1,1.8937+0.4796)\ =0.58834$

and $k_3=hf((x_1+h/2,y_1+k_2/2)=(0.1)f(1.5,1.8937+0.58743)$ $=(0.1)[(1.05)^2+(1.6933)^2]$

$=0.611715$

$k_4=hf(x_1+h,y_1+k_3)=(0.1)f(1.2,1.8937+0.610728)$

$=0.77261$

Hence $y_2=1.8937+1/6(0.4796+2(0.58834)+2(0.611715)+0.7726)\ =2.5043$

$\therefore$ y =2.5043 where $x = 0.2$

4. Using R-K method, find y(0.2) for the eqn dy/dx=y-x,y(0)=1,take h=0.2

Ans:1.15607

5.Given that y^1=y-x,y(0)=2 find y(0.2) using R- K method take h=0.1

Ans: 2.4214

6. Apply the 4th order R-K method to find $y(0.2)$ *and* $y(0.4)$ for one equation $10\frac{dy}{dx} = x^2 + y^2, y(0) = 1$ take h = 0.1 Ans. 1.0207, 1.038

7. using R-K method, estimate y(0.2) and y(0.4) for the eqn dy/dx=y^2-x^2/ y^2+x^2,y(0)=1,h=0.2

Ans:1.19598,1.3751

8. use R-K method, to approximate y when x=0.2 given that y^1=x+y,y(0)=1

Sol: Here f(x,y)=x+y,y_0=1,x_0=0

Since h is not given for better approximation of y

Take h=0.1

$\therefore$ x_1=0.1, x_2=0.2

Step1

To find y_1 i.e y(x_1)=y(0.1)

By R-K method,we have

y_1=y_0+1/6 (k_1+2k_2+2k_3+k_4)

Where k_1=hf(x_0,y_0)=(0.1)f(0,1)=(0.1) (1)=0.1

k_2= hf (x_0+h/2,y_0+k_1/2)=(0.1)f(0.05,1.05)=0.11

and k_3=hf((x_0+h/2,y_0+k_2/2)=(0.1)f(0.05,1+0. 11/2)=(0.1)[(0.05) +(4.0.11/2)]

=0.1105

k_4=h f (x_0+h,y_0+k_3)=(0.1)f(0.1,1.1105)=(0.1)[0.1+1.1105]

=0.12105

Hence $\therefore\ y_1 = y(0.1) = 1 + \frac{1}{6}(0.1 + 0.22 + 0.240 + 0.12105)$

y = 1.11034

Step2:

To find y_2 i.e y(x_2) = y(0.2)

Here x_1=0-1, y_1=1.11034 and h=0.1

Again By R-K method,we have

y_2=y_1+1/6(k_1+2k_2+2k_3+k_4)

k_1=h f(x_1,y_1)=(0.1)f(0.1,1.11034)=(0.1) [1.21034]=0.121034

k_2= h f (x_1+h/2, y_1+k_1/2)=(0.1)f(0.1+0.1/2,1.11034+0.121034/2)

=0.1320857

and k_3=h f((x_1+h/2,y_1+k_2/2)=(0.1)f(0.15,1.11034+0.1320857/2)

=0.1326382

k_4=h f(x_1+h,y_1+k_3)=(0.1)f(0.2,1.11034+0.1326382)

(0.1)(0.2+1.2429783)=0.1442978

Hence y_2=1.11034+1/6(0.121034+0.2641714+0.2652764+0.1442978

=1.11034+0.1324631 =1.242803

$\therefore$ y =1.242803 when x=0.2

9.Using Runge-kutta method of order 4,compute y(1.1) for the eqn y^1=3x+y^2,y(1)=1.2 h = 0.05